<u>Theorem 3.5.23</u> The eigenvectors corresponding to two distinct eigen values of a real symmetric matrix are orthogonal.

Proof: Let A be a real symmetric matrix. Let X_1, X_2 be two eigenvectors of A corresponding to two distinct eigen values λ_1 and λ_2

Then $AX_1 = \lambda_1 X_1$ and $AX_2 = \lambda_2 X_2$

Now $AX_1 = \lambda_1 X_1 \Rightarrow (AX_1)^t = (\lambda_1 X_1)^t = \lambda_1 X_1^t$ as λ_1 is real. or, $X_1^t A = \lambda_1 X_1^t$ as $A^t = A$.

Multiplying by X_2 from the right, we have

$$X_1^t A X_2 = \lambda_1 X_1^t X_2$$

$$\text{or, } X_1^t (\lambda_2 X_2) = \lambda_1 X_1^t X_2$$

$$\text{or, } (\lambda_2 - \lambda_1) X_1^t X_2 = 0$$

As $\lambda_1 \neq \lambda_2$, so $X_1^t X_2 = 0$. Since $X_1 \neq 0$, $X_2 \neq 0$, it follows that X_1 is orthogonal to X_2

<u>Theorem 3.5.24</u> Each eigen value of a real orthogonal matrix has unit modulus

Proof: Let A be an $n \times n$ real orthogonal matrix. Then $A^t A = A A^t = I_n$. The eigen values of A are in general, complex numbers, some of which may be purely real. Let λ be an eigen value of A. Then $\det(A - \lambda I_n) = 0$

So, $\exists$ a non-null solution of the homogeneous system $(A-\lambda I_n)X=0$. Let X_1 be one such solution. Then $(A-\lambda I_n)X_1=0$ or, $AX_1=\lambda X_1$

Taking transpose of the conjugate, we have

$$(\overline{AX_1})^t = (\overline{\lambda X_1})^t$$

$$\text{or, } (\overline{X_1})^t(\overline{A})^t = \bar{\lambda}(\overline{X_1})^t$$

$$\text{or, } (\overline{X_1})^t A^t = \bar{\lambda}(\overline{X_1})^t \quad \text{since } (\overline{A})^t = A^t$$

Multiplying AX_1 from the right, we have

$$(\overline{X_1})^t A^t (AX_1) = \bar{\lambda}(\overline{X_1})^t(AX_1)$$

$$\text{or, } (\overline{X_1})^t(A^tA)X_1 = \bar{\lambda}(\overline{X_1})^t\lambda X_1$$

$$\text{or, } (\overline{X_1})^t X_1 = \lambda\bar{\lambda}(\overline{X_1})^t X_1 \quad \text{since } A^tA = I_n$$

This implies $(1-\lambda\bar{\lambda})(\overline{X_1})^t X_1 = 0$

Since X_1 is non-null, $(\overline{X_1})^t X_1 \neq 0$,

it follows that $\lambda\bar{\lambda}=1$ or, $|\lambda|=1$

This proves the theorem.

Worked Examples

1. If λ be an eigenvalue of a real orthogonal matrix A then $\frac{1}{\lambda}$ is an eigen value of A.

Solution: Let A be an orthogonal matrix. Then $AA^t = A^tA = I_n$ and A is non-singular.

Since A is non-singular, $\lambda \neq 0$. Since λ is an eigen value of A, let $(A-\lambda I_n)=0$

or, $\det(A - \lambda A A^t) = 0$

or, $\det A \det(I_n - \lambda A^t) = 0$

or, $\det(I_n - \lambda A^t) = 0$ as $\det A \neq 0$

or, $(-1)^n \lambda^n \det(A^t - \frac{1}{\lambda} I_n) = 0$

or, $\det(A^t - \frac{1}{\lambda} I_n) = 0$ as $\lambda \neq 0$

or, $\det(A - \frac{1}{\lambda} I_n)^t = 0$ as $(A^t - \frac{1}{\lambda} I_n) = (A - \frac{1}{\lambda} I_n)^t$

or, $\det(A - \frac{1}{\lambda} I_n) = 0$

So, $\frac{1}{\lambda}$ is an eigen value of A.

2. A is a 3×3 real matrix having the eigen value $2, 3, 1$. $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ are the eigen vectors of A corresponding to the eigen values $2, 3, 1$ respectively. Find the matrix A.

Solution. Let $X_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Then $AX_1 = 2X_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$, $AX_2 = 3X_2 = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$

and $AX_3 = X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Let $P = (X_1 \; X_2 \; X_3) = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Then $AP = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 3 & 1 \\ 2 & 3 & 1 \end{pmatrix}$ As X_1, X_2, X_3 are eigen vectors corresponding to distinct eigen values $2, 3, 1$, $\{X_1, X_2, X_3\}$ is linearly independent. Hence P is non-singular.

$$P^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} \quad \text{So, } A = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 3 & 1 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 3 & 0 \\ -2 & 1 & 2 \end{pmatrix}$$

3. If S be a real skew symmetric matrix of order n, prove that (i) $I_n + S$ is non-singular

(ii) $(I_n+S)^{-1}(I_n-S)$ is orthogonal.

(iii) if X be an eigen vector of S with eigen value λ then X is also an eigen vector of the matrix $(I_n+S)^{-1}(I_n-S)$ with eigen value $\frac{1-\lambda}{1+\lambda}$,

(iv) if $\bar{S} = (I_n+S)^{-1}(I_n-S)$ then ~~&~~ $I_n+\bar{S}$ is also non-singular and $\bar{\bar{S}} = S$

Solution : (i) Since S is a real skew symmetric matrix, its eigen values are imaginary or zero. Hence -1 is not an eigen value of S. So $\det(S-(-1)I_n) \neq 0$

So, $\det(I_n+S) \neq 0$. Hence I_n+S is non-singular.

(ii) Let $P = (I_n+S)^{-1}(I_n-S)$

Then $PP^t = (I_n+S)^{-1}(I_n-S)\left[(I_n+S)^{-1}(I_n-S)\right]^t$

$= (I_n+S)^{-1}(I_n-S)(I_n-S)^t\left((I_n+S)^{-1}\right)^t$

$= (I_n+S)^{-1}(I_n-S)(I_n-S)^t\left((I_n+S)^t\right)^{-1}$

$= (I_n+S)^{-1}(I_n-S)(I_n+S)(I_n-S)^{-1}$

as $S^t = -S$

$= (I_n+S)^{-1}(I_n+S)(I_n-S)(I_n-S)^{-1}$

(as $(I_n+S)(I_n-S) = (I_n-S)(I_n+S)$)

$= I$. So, P is orthogonal.

(iii) $SX = \lambda X$. Therefore $(I_n+S)^{-1}(I_n-S)X =$

$= (I_n+S)^{-1}(1-\lambda)X = (1-\lambda)(I_n+S)^{-1}X$

Again $(I_n+S)X = (1+\lambda)X$

or, $X = (I_n+S)^{-1}(1+\lambda)X = (1+\lambda)(I_n+S)^{-1}X$

So, we have $\frac{1}{1+\lambda}X = (I_n+S)^{-1}X$ since $\lambda+1 \neq 0$

So, $(I_n+S)^{-1}(I_n-S)X = (1-\lambda)\frac{1}{1+\lambda}X = \frac{1-\lambda}{1+\lambda}X$

This proves that X is an eigen vector of $(I_n+S)^{-1}(I_n-S)$ with eigen value $\frac{1-\lambda}{1+\lambda}$.

(iv) $\bar{S} = (I_n+S)^{-1}(I_n-S)$

$$I_n+\bar{S} = (I_n+S)^{-1}(I_n+S) + (I_n+S)^{-1}(I_n-S)$$
$$= (I_n+S)^{-1}(I_n+S+I_n-S)$$
$$= 2(I_n+S)^{-1}$$

Therefore, $(I_n+\bar{S})^{-1} = \frac{1}{2}(I_n+S)$, proving that $I_n+\bar{S}$ is non-singular.

Again, $I_n-\bar{S} = (I_n+S)^{-1}(I_n+S) - (I_n+S)^{-1}(I_n-S)$
$$= (I_n+S)^{-1}(I_n+S-I_n+S)$$
$$= 2(I_n+S)^{-1}S$$

So, $\bar{\bar{S}} = (I_n+\bar{S})^{-1}(I_n-\bar{S}) = (I_n+S)(I_n+S)^{-1}S$
$$= S$$

——— x ——— END