Notes on Unit 2 (Differentiability of functions) of Theory of real functions (Core Course V) by SB (Subhanandar Bandyopadhyay)

Books followed: 1. Introduction to Real Analysis – S.K. Mapa
2. Principles of Mathematical Analysis – Walter Rudin.
3. Real Analysis – John M. Howie
4. Mathematical Analysis – S.C. Malik & S. Arora

1.1.1 Differentiability (or Derivability) at a point : (Definition) :

Let f be a real valued function defined on an interval $I = [a,b] \subseteq \mathbb{R}$. f is said to be differentiable (or derivable) at an interior point c (where $a < c < b$) if $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ or $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists.

The limit, in case it exists, is called the differential coefficient (or derivative) of the function f at $x = c$ and is denoted by $f'(c)$. The limit exists when the left hand and right limit exists and are equal.

$\lim_{x \to c-} \frac{f(x) - f(c)}{x - c}$, if exists, is called the left hand derivative of f at c and is denoted by $f'(c-)$ or $Lf'(c)$

Similarly, $\lim_{x \to c+} \frac{f(x) - f(c)}{x - c}$, if exists, is called the right hand derivative of f at c and is denoted by $f'(c+)$ or $Rf'(c)$.

1.1.2 Definition (Differentiability (or derivability) of a function in an interval) :

A real valued function defined on $[a,b]$ is said to be differentiable (or derivable) at the end point a, i.e., $f'(a)$ exists if

$$\lim_{x \to a+} \frac{f(x) - f(a)}{x - a} \text{ exists.}$$

In other words, $f'(a) = \lim_{x \to a+} \frac{f(x) - f(a)}{x - a}$

Similarly, f is said to be differentiable (or derivable) at the end point b, if $\lim_{x \to b-} \frac{f(x) - f(b)}{x - b}$ exists.

~~f is said to be differentiable~~. If f is differentiable (or, derivable) at all points in the interval $[a,b]$ except at the end points, f is said to be differentiable (or derivable) in the open interval (a,b).

f is said to be differentiable (or derivable) in the closed interval $[a,b]$, if it is differentiable (or derivable) in the open interval (a,b) and also at the end points a and b.

Theorem 1.1.3 Let I be an interval and a function $f: I \to \mathbb{R}$ be differentiable at a point $c \in I$. Then f is continuous at c.

Proof: For all $x \in I$ but $x \neq c$, $f(x) - f(c) = \frac{f(x)-f(c)}{(x-c)} \cdot (x-c)$

Also $\lim_{x\to c}(x-c) = 0$. Now $\lim_{x\to c}\left(f(x)-f(c)\right) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c} \cdot \lim_{x\to c}(x-c)$

$= f'(c)\cdot 0 = 0$, since $f'(c)$ is finite

So, $\lim_{x\to c} f(x) = f(c)$ and this shows that f is continuous at c.

Note: The converse is not in general true; i.e., the continuity of f at a point $c \in I$ does not ensure the differentiability of f at c.

For example, let $f(x) = |x|$, $x \in \mathbb{R}$.

At $x=0$, $f(x) = 0$. Also f is continuous at 0.

$$\lim_{x\to 0+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0+}\frac{x-0}{x-0} = 1$$

$$\lim_{x\to 0-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0-}\frac{-x-0}{x-0} = -1$$

As $Rf'(0) \neq Lf'(0)$, f is not differentiable at 0.

Let I be an interval and $f: I \to \mathbb{R}$ is differentiable at a point $c \in I$. Then $f'(c)$ exists. Let A be a subset of I such at every point of A, f is differentiable. Then $f'(c)$ exists for each $c \in A$.

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f' can be considered as a function on A. f' is said to be the derived function of f on A.

Examples 1. Let $k \in \mathbb{R}$ and $f(x) = k$, $x \in \mathbb{R}$. Find the derived function f' and its domain.

Solution: Let $c \in \mathbb{R}$, When $x \neq c$ $\frac{f(x)-f(c)}{x-c} = \frac{k-k}{x-c} = 0$

So, $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = 0$. So, $f'(c) = 0$ for all $c \in \mathbb{R}$

So the derived function f' is defined by $f'(x) = 0$, $x \in \mathbb{R}$.

The domain of f' is $\mathbb{R}$.

2. Let $f(x) = x^2$, $x \in \mathbb{R}$. Find the derived function f' and its domain

Solution: Let $c \in \mathbb{R}$. Then $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c}(x+c)$ (as $x \neq c$)

$= 2c$

So, $f'(c) = 2c$ for all $c \in \mathbb{R}$

So, The derived function f' is defined by

$f'(x) = 2x$, $x \in \mathbb{R}$. The domain of f' is $\mathbb{R}$.

3. Let $f(x) = \sqrt{x}$ $x \in [0, \infty)$. Find the derived function f' and its domain. Let $c \in [0, \infty)$

$\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$

$= \lim_{x\to c} \frac{\sqrt{x}-\sqrt{c}}{x-c} = \lim_{x\to c} \frac{1}{\sqrt{x}+\sqrt{c}} = \frac{1}{2\sqrt{c}}$ if $c \neq 0$

So, $f'(c) = \frac{1}{2\sqrt{c}}$ if $c \in (0, \infty)$

So the domain of f' is $(0, \infty)$ and f' is defined by $f(x) = \frac{1}{2\sqrt{x}}$, $x \in (0, \infty)$

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1. Let $f : [0, 3] \to \mathbb{R}$ be defined by

$$f(x) = x, \quad 0 \le x \le 1$$
$$= 2 - x^2, \quad 1 < x < 2$$
$$= x - x^2, \quad 2 \le x \le 3$$

Find the derived function f' and its domain.

Solution:

$$f'(x) = 1 \text{ for } 0 < x < 1 \text{ or for } x \in (0,1)$$
$$= -2x \text{ for } 1 < x < 2 \text{ or for } x \in (1,2)$$
$$= 1 - 2x \text{ for } 2 < x < 3 \text{ or for } x \, (2,3)$$

$\lim\limits_{x \to 0+} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0+} \dfrac{x - 0}{x - 0} = 1$. So, $Rf'(0) = 1$

Hence f is differentiable at 0 and $f'(0) = 1$

$\lim\limits_{x \to 1-} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1-} \dfrac{x - 1}{x - 1} = 1$. So, $Lf'(1) = 1$

$\lim\limits_{x \to 1+} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1+} \dfrac{2 - x^2 - 1}{x - 1} = \lim\limits_{x \to 1+} \dfrac{1 - x^2}{x - 1} = \lim\limits_{x \to 1+} -(1 + x) = -2$

So, $Rf'(1) = -2$ So, $Lf'(1) \ne Rf'(1)$. So, f is not differentiable at 1.

$\lim\limits_{x \to 2-} \dfrac{f(x) - f(2)}{x - 2} = \lim\limits_{x \to 2-} \dfrac{2 - x^2 - (-2)}{x - 2} = \lim\limits_{x \to 2-} \dfrac{4 - x^2}{x - 2} = \lim\limits_{x \to 2-} -(2 + x) = -4$

So, $Lf'(2) = -4$

$\lim\limits_{x \to 2+} \dfrac{f(x) - f(2)}{x - 2} = \lim\limits_{x \to 2+} \dfrac{x - x^2 - (-2)}{x - 2} = \lim\limits_{x \to 2+} \dfrac{-(x^2 - x - 2)}{x - 2} = \lim\limits_{x \to 2+} -(x + 1) = -3$

So, $Rf'(2) = -3$. So, $Lf'(2) \ne Rf'(2)$. So f is not differentiable at 2.

$\lim\limits_{x \to 3-} \dfrac{f(x) - f(3)}{x - 3} = \lim\limits_{x \to 3-} \dfrac{x - x^2 - (-6)}{x - 3} = \lim\limits_{x \to 3-} \dfrac{-(x^2 - x - 6)}{x - 3}$
$= \lim\limits_{x \to 3-} \dfrac{-(x - 3)(x + 2)}{x - 3} = -5$. So, $Rf'(3) = -5$.

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