Since for all $n \in \mathbb{N}$, $f^n(x)$ exists for all $x > -1$, the right hand side polynomial in (i) takes the form of an infinite series as $n \to \infty$. The infinite series will converges to $f(x)$ for those non-zero $x > -1$ for which $\lim_{n \to \infty} R = 0$.

$$R_n = \frac{x^n (1-\theta)^{n-1}}{(n-1)!} m(m-1)\cdots(m-n+1)(1+\theta x)^{m-n}$$

$$= \frac{m(m-1)\cdots(m-n+1)}{(n-1)!} x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \cdot (1+\theta x)^{m-1}.$$

Let $0 < |x| < 1$

Let $u_n = \left| \frac{m(m-1)\cdots(m-n+1)}{(n-1)!} \right| |x|^n$

Then $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = |x| < 1$. So, $\lim_{n \to \infty} u_n = 0$ ... (ii)

If $0 < |x| < 1$, then $0 < 1 - \theta < 1 + \theta x < 1 + \theta$.

So, $0 < \frac{1-\theta}{1+\theta x} < 1$ and hence $0 < \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} < 1$ ... (iii)

For all real x, $-|x| \le x \le |x|$.

Hence $-|x| \le -\theta|x| \le \theta x \le \theta |x| < |x|$ since $0 < \theta < 1$.

So, $0 < |x| < 1$, $0 < 1 - |x| < 1 + \theta x < 1 + |x|$ and therefore

$$(1+\theta x)^{m-1} < (1+|x|)^{m-1} < 2^{m-1} \quad \text{if } m > 1$$

$$\text{and } (1+\theta x)^{m-1} < (1-|x|)^{m-1} \quad \text{if } m < 1$$

Thus for all $m (\ne 1)$, $(1+\theta x)^{m-1}$ is bounded ... (iv)

From (ii), (iii) and (iv) it follows that $\lim_{n \to \infty} |R_n| = 0$, when $0 < |x| < 1$ and so, $\lim_{n \to \infty} R_n = 0$, when $0 < |x| < 1$

The series $1+mx+\frac{m(m-1)}{2!}x^2+\dots+\frac{m(m-1)\dots(m-n+1)}{n!}x^n+\dots$

converges to $(1+x)^m$ for all non-zero $x \in (-1,1)$

At $x=0$, the convergence holds trivially.

So, $(1+x)^m = 1+mx+\frac{m(m-1)}{2!}x^2+\dots+\frac{m(m-1)\dots(m-n+1)}{n!}x^n+\dots$

for all $x \in (-1,1)$.

2.2 Indeterminate forms

In discussion on limits that we have learned earlier, it was shown that if $\lim_{x\to c} f(x) = l$ and $\lim_{x\to c} g(x) = m \neq 0$ then $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{l}{m}$. If however, $m=0$, then the limit could not be evaluated. The case when $l=0$ and $m=0$ was not ~~cover~~ covered earlier. In this case the limit of the quotient $\frac{f}{g}$ is said to take the indeterminate form $\frac{0}{0}$. ~~The other~~ The other indeterminate forms are represented by the symbols $\frac{\infty}{\infty}$, $\infty-\infty$, 0^0, 1^∞, $1^{-\infty}$, ∞^0.

We now state several theorems (without proof) concerning evaluation of indeterminate forms: Indeterminate form, $0/0$

Theorem 2.2.1 If f, g be two functions such that

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and

(ii) $f'(a), g'(a)$ exist and $g'(a) \neq 0$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Note: Condition (i) can be replaced by $f(a) = g(a) = 0$

Theorem 2.2.2 (L'Hospital's Rule for $0/0$ form): If f, g are two functions such that (i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

(ii) $f'(x), g'(x)$ exists, and $g'(x) \neq 0$, $\forall x \in (a-\delta, a+\delta)$, $\delta > 0$ except possibly at a, and

(iii) $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists,

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Theorem 2.2.3 (Generalised L'Hospital's Rule for 0/0 forms)

If f, g be two functions such that

(i) $f^{r}(x), g^{r}(x)$ exist, and $g^{r}(x) \neq 0$ $(r = 0, 1, 2, \dots, n)$ for any x in $(a-\delta, a+\delta)$, $\delta > 0$ except possibly at $x = a$

(ii) $\lim_{x \to a} f(x) = \lim_{x \to a} f'(x) = \dots = \lim_{x \to a} f^{n-1}(x) = 0$

and $\lim_{x \to a} g(x) = \lim_{x \to a} g'(x) = \dots = \lim_{x \to a} g^{n-1}(x) = 0$

and (iii) $\lim_{x \to a} \frac{f^{n}(x)}{g^{n}(x)}$ exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^{n}(x)}{g^{n}(x)}$$

Theorem 2.2.4 If f, g be two functions such that

(i) $\lim_{x \to a} f(x) = \lim_{x \to a} f'(x) = \dots = \lim_{x \to a} f^{n-1}(x) = 0$

and $\lim_{x \to a} g(x) = \lim_{x \to a} g'(x) = \dots = \lim_{x \to a} g^{n-1}(x) = 0$

and (ii) $f^{n}(a), g^{n}(a)$ exist and $g^{n}(a) \neq 0$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^{n}(a)}{g^{n}(a)}$$

Example 1 If f' exists in the neighbourhood of $x = a$ and $f''(a)$ exist, show that $\lim_{h \to 0} \frac{f(a+2h) - 2f(a+h) + f(a)}{h^2}$ exists and is equal to $f''(a)$

Solution: $\lim_{h\to 0} \frac{f'(a+2h) - f'(a)}{2h} = \lim_{h\to 0} \frac{f'(a+h) - f'(a)}{h} = f''(a)$

Hence $\lim_{h\to 0} \frac{2f'(a+2h) - 2f'(a+h)}{2h} = \cancel{\lim_{h\to 0}\left(2\left(\frac{f'(a+h) - f'(a)}{2h}\right)\right)}$

$$= \lim_{h\to 0} \frac{f'(a+2h) - f'(a+h)}{h}$$

$$= \lim_{h\to 0}\left(2\left(\frac{f'(a+2h) - f'(a)}{2h}\right) - \left(\frac{f'(a+h) - f'(a)}{h}\right)\right)$$

$$= 2\lim_{h\to 0} \frac{f'(a+2h) - f'(a)}{2h} - \lim_{h\to 0}\left(\frac{f'(a+h) - f'(a)}{h}\right)$$

$$= 2f''(a) - f''(a) = f''(a)$$

So, by Theorem 2.2.3,

$$\lim_{h\to 0}\left(\frac{f(a+2h) - 2f(a+h) + f(a)}{h^2}\right) = \lim_{h\to 0} \frac{2f'(a+2h) - 2f'(a+h)}{2h}$$
$$= f''(a)$$

Theorem 2.2.5 (L'Hospital rule for infinite limits) If f, g be two functions such that

(i) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$

(ii) $f'(x), g'(x)$ exist and $g'(x) \neq 0$, ~~for x>0~~ $\forall x$ in $[0, \infty)$

and (iii) $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

Indeterminate form, ∞/∞.

If $f(x)$ and $g(x)$ both tend to ∞ as $x\to a$ then $\lim_{x\to a} \frac{f(x)}{g(x)}$ takes ∞/∞ form.