Theorem 2.2.6 (L'Hospital's rule for ∞/∞ form) If f, g be two functions such that

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$

(ii) $f'(x), g'(x)$ exist and $g'(x) \neq 0$, $\forall x$ in $(a-\delta, a+\delta)$, $\delta > 0$ except possibly at a, and

(iii) $\lim_{x\to a} \frac{f'(x)}{g'(x)} = l$

then $\lim_{x\to a} \frac{f(x)}{g(x)} = l$

2.3 (a) Form $0 \times \infty$

when $f(x) \to 0$ and $g(x) \to \infty$ as $x \to a$, $f(x).g(x)$ takes $0\times\infty$ form. However $f(x).g(x)$ may be expressed as

$$\frac{f(x)}{1/g(x)} \quad \text{or} \quad \frac{g(x)}{1/f(x)}$$ which has respectively

$0/0$ and $\frac{\infty}{\infty}$ forms

(b) Form $\infty - \infty$

This can be reduced to the form $0/0$ or ∞/∞, for

$$f(x) - g(x) = \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)\,g(x)}} \quad \left(\frac{0}{0} \text{ form}\right)$$

(c) Form 0^0, 1^∞, ∞^0

These forms can be made to depend upon the previous form by putting $z = (f(x))^{g(x)}$, so that $\log z = g(x).\log(f(x))$. Then it is of the form $0/0$ or $\frac{\infty}{\infty}$.

So, we find $\lim \log z$ and as $\lim \log z = \log \lim z$ we get $\lim z$ =

Let us now evaluate some limits which take up these above mentioned forms. We shall use the known limits, like $\lim_{x \to 0} \frac{\sin x}{x}$, $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ etc.

Example 2 Let $f(x) = x^2 \sin \frac{1}{x}$, $x \neq 0$ and $g(x) = \sin x$, $x \in \mathbb{R}$
$= 0$, $x = 0$

We try to find $\lim_{x \to 0} \frac{f(x)}{g(x)}$.

Here $f(0) = g(0) = 0$, $g(x) \neq 0$ in some deleted neighbourhood of 0, $f'(0)$ and $g'(0)$ both exist and $g'(0) = 1 \neq 0$

So, $\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)} = 0$

Example 3 Let $f(x) = \sin x$, $x \in \mathbb{R}$, $g(x) = \sqrt{x}$, $x \in [0, \infty)$
To find $\lim_{x \to 0} \frac{f(x)}{g(x)}$, we see that L'Hospital's rule can not be applied here as $g'(0)$ does not exist.

Example 4 Evaluate the limit
$$\lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$$
Here the limit takes $0/0$ form. We use L'Hospital's rule successively. The evaluation of the limit can be exhibited as follows:

$$\lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{e^x + e^{-x} - \frac{2}{1+x}}{x \cos x + \sin x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} + \frac{2}{(1+x)^2}}{-x \sin x + 2\cos x} = 1$$

Example 5 Evaluate $\lim_{x\to 0} \frac{x-\tan x}{x^3}$

Let $f(x) = x - \tan x$ and $g(x) = x^3$

$\lim_{x\to 0} f(x) = 0$ $\lim_{x\to 0} g(x) = 0$

Hence $\lim_{x\to 0} \frac{f(x)}{g(x)}$ is of $\frac{0}{0}$ form. So, Using L'Hospital's

Rule $\lim_{x\to 0} \frac{x-\tan x}{x^3} = \lim_{x\to 0} \frac{1-\sec^2 x}{3x^2}$

$= -\frac{1}{3} \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^2 = -\frac{1}{3}$

Example 6 Evaluate $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$

Solution: It is a $0/0$ form and so

$$\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2} = \lim_{x\to 0} \frac{xe^x + e^x - \frac{1}{1+x}}{2x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{xe^x + 2e^x + \frac{1}{(1+x)^2}}{2}$$

$$= \frac{3}{2}$$

Example 7 Evaluate $\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x}$

Solution: Since $\lim_{x\to 0} (1+x)^{1/x} = e$, therefore it is a $0/0$ form

$$\text{So,} \quad \lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x} = \lim_{x\to 0} \frac{(1+x)^{1/x}\{x - (1+x)\log(1+x)\}}{x^2(1+x)}$$

$$= e. \lim_{x\to 0} \frac{x - (1+x)\log(1+x)}{x^2(1+x)} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= e. \lim_{x\to 0} \frac{-\log(1+x)}{2x+3x^2} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= e.\lim_{x\to 0}\frac{-1}{(2+6x)(1+x)} = -\frac{e}{2}$$

Example 8 Find $\lim_{x\to 1-}\frac{\log(1-x)}{\cot(\pi x)}$

Solution : It is ∞/∞ form and so,

$$\lim_{x\to 1-}\frac{\log(1-x)}{\cot(\pi x)} = \cancel{\frac{\frac{-1}{1-x}}{-\pi\cos^2(\pi x)}}$$

$$= \lim_{x\to 1-}\frac{-\frac{1}{1-x}}{-\pi\,\mathrm{cosec}^2(\pi x)} = \lim_{x\to 1-}\frac{\sin^2\pi x}{\pi(1-x)} \quad [0/0 \text{ form}]$$

$$= \lim_{x\to 1-} \cancel{2\pi\sin 2\pi x}\ \frac{\pi\sin(2\pi x)}{-\pi} = 0$$

Example 9 Evaluate $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$

Solution: It is a $\infty-\infty$ form, we therefore write it as

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right) = \lim_{x\to 0}\frac{\sin^2 x - x^2}{x^2\sin^2 x} \quad [0/0 \text{ form}]$$

$$= \cancel{\lim_{x\to 0}\frac{\sin 2x - 2x}{2x\sin^2 x + x^2\sin 2x}} = \lim_{x\to 0}\frac{\sin 2x - 2x}{2x\sin^2 x + x^2\sin 2x} \quad [0/0 \text{ form}]$$

$$= \lim_{x\to 0}\frac{2\cos 2x - 2}{2\sin^2 x + 2x\sin 2x + 2x\sin 2x + 2x^2\cos 2x}$$

$$= \lim_{x\to 0}\frac{\cos 2x - 1}{\sin^2 x + 2x\sin 2x + x^2\cos 2x} \quad [0/0 \text{ form}]$$

$$= \lim_{x\to 0}\frac{-2\sin 2x}{3\sin 2x + 6x\cos 2x - 2x^2\sin 2x} \quad [0/0 \text{ form}]$$

$$= \lim_{x\to 0}\frac{-4\cos 2x}{6\cos 2x + 6\cos 2x - 12x\sin 2x - 4x\sin 2x - 4x^2\cos 2x}$$

$$= \frac{-4}{12} = -\frac{1}{3}$$