Example 10 Evaluate $\lim\limits_{x\to 0+} (\sin x \log x)$

Solution: It is a $0\times\infty$ form. Let us write

$$\lim_{x\to 0+}(\sin x\log x) = \lim_{x\to 0+}\frac{\log x}{\operatorname{cosec} x} \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= -\lim_{x\to 0+}\frac{1/x}{\operatorname{cosec} x\cot x} = -\lim_{x\to 0+}\left(\frac{\sin x}{x}\right)\tan x = 0$$

Example 11 Evaluate $\lim\limits_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$

Solution: It is a 1^∞ form. So,

let $y = \left(\frac{\tan x}{x}\right)^{1/x^2}$. So, $\log y = \frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$

So, $\lim\limits_{x\to 0}\log y = \lim\limits_{x\to 0}\dfrac{\log\left(\frac{\tan x}{x}\right)}{x^2}$ $\left[\frac{0}{0}\text{ form}\right]$

$$= \lim_{x\to 0}\frac{\frac{\sec^2 x}{\tan x} - \frac{1}{x}}{2x} = \lim_{x\to 0}\frac{x\sec^2 x - \tan x}{2x^2\tan x} \quad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{\sec^2 x + 2x\sec^2 x\tan x - \sec^2 x}{4x\tan x + 2x^2\sec^2 x}$$

$$= \lim_{x\to 0}\frac{\sec^2 x\tan x}{2\tan x + x\sec^2 x}$$

$$= \lim_{x\to 0}\frac{\tan x}{\sin 2x + x} \quad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{\sec^2 x}{2\cos 2x + 1} = \frac{1}{3}$$

So, $\lim\limits_{x\to 0}\log y = \frac{1}{3}$

or, $\log\lim\limits_{x\to 0} y = \frac{1}{3}$

or, $\lim\limits_{x\to 0} y = e^{1/3}$

So, $\lim\limits_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2} = e^{1/3}$

Exercises Evaluate the following limits :

1. $\lim_{x\to 0} \frac{x-\sin x}{x^3}$ 2. $\lim_{x\to 0} \frac{x-\log(1+x)}{1-\cos x}$

3. $\lim_{x\to 0} \frac{\log(1+x^3)}{\sin^3 x}$ 4. $\lim_{x\to\infty} \frac{x^3}{e^x}$

5. $\lim_{x\to \pi/2+} \frac{\log(x-\pi/2)}{\tan x}$ 6. $\lim_{x\to\infty} x\tan\frac{1}{x}$

7. $\lim_{x\to 0}\left(\cot^2 x-\frac{1}{x^2}\right)$ 8. $\lim_{x\to\pi/2}(\sec x-\tan x)$

9. $\lim_{x\to 0+}(\cot x)^{\sin x}$ 10. $\lim_{x\to\pi/2}(\sin x)^{\tan x}$

11. $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x}$ 12. $\lim_{x\to 0}\frac{e^x-2\cos x+e^{-x}}{x\sin x}$

13. $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$ 14. $\lim_{x\to 1} x^{1/(x-1)}$

15. Find the values of a and b in order that ~~limit~~ $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}$ may be equal to 1

16. Find the value of a and the limit in order that $\lim_{x\to 0}\frac{\sin 2x+a\sin x}{x^3}$ be finite.

2.3 Maxima and Minima

Let I be an interval.

A function $f: I\to\mathbb{R}$ is said to have a local maximum (or a relative maximum) at a point $c\in I$ if there exists neighbourhood $N(c, \delta)$ of c such that $f(c) \geq f(x)$ for all $x \in N(c, \delta) \cap I$.

f is said to have a local minimum (or a relative minimum) at a point $c \in I$ if there exists a neighbourhood $N(c, \delta)$ of c such that $f(c) \leq f(x)$ for all $x \in N(c, \delta) \cap I$.

We say that f has a local extremum (or a ~~re~~ relative extremum) at a point $c \in I$ if f has either a local maximum or a local minimum at c.

<u>Theorem 2.3.1</u> Let $f : I \to \mathbb{R}$ be such that f has a local extremum at an interior point c of I. If $f'(c)$ exists then $f'(c) = 0$

Proof: We prove the theorem for the case when f has a local maximum at c. The proof for the other case is similar.

Since $f'(c)$ exists, either $f'(c) > 0$, or $f'(c) < 0$ or $f'(c) = 0$.

Let $f'(c) > 0$. Then $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} > 0$.

Therefore there exists a positive δ such that $\dfrac{f(x) - f(c)}{x - c} > 0$ for all $x \in N'(c, \delta) \subset I$ where $N'(c, \delta) = (c - \delta, c + \delta) - \{c\}$

Let $c < x < c + \delta$. Then $x - c > 0$ and therefore $f(x) > f(c)$ for all $x \in (c, c+\delta)$. This contradicts that f has a local extremum at c. Consequently, $f'(c) \not> 0$ ... (i)

Let $f'(c) < 0$. Then $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} < 0$. Therefore there exists a positive δ such that $\dfrac{f(x) - f(c)}{x - c} < 0$

for all $x \in N'(c,\delta) \subset I$ where $N'(c,\delta) = (c-\delta, c+\delta) - \{c\}$

Let $c-\delta < x < c$. Then $x-c<0$ and therefore $f(x) > f(c)$ for all $x \in (c-\delta, c)$. This contradicts that f has a local maximum at c.

Consequently, $f'(c) \not< 0$ ... (ii)

From (i) and (ii) and ~~law of trichotomy of Real numbers~~ we have $f'(c)=0$

This completes the proof.

Corollary 2.3.2 Let $f: I \to \mathbb{R}$ and c be an interior point of I, where f has a local extremum. Then either $f'(c)$ does not exists or $f'(c)=0$

Note 1 The theorem says that if the derivative $f'(c)$ exists at an interior point c of local extremum, $f'(c)$ must be 0. A function may, however have a local extremum at an interior point c of its domain without being differentiable at c. For example, the function defined by $f(x) = |x|$, $x \in \mathbb{R}$ has a local minimum at 0 but $f'(0)$ does not exists.

Note 2 The condition $f'(c)=0$ (when $f'(c)$ exists) is only a necessary condition for an interior point c to be a point of local extremum of the function f.

For examples the function f defined by $f(x) = x^3$, $x \in \mathbb{R}$, 0 is an interior point of the domain of f. $f'(0)=0$ but 0