is neither a point of local maximum nor a point of local minimum for the function $f$.

Note 3 The theorem holds if $c$ is an interior point of $I$. Then $f$ has a local maximum at 1 (not an interior point of $[0,1]$. $f$ is differentiable ~~and~~ at 1, but $f'(1) \neq 0$.

Theorem 2.3.2 (First derivative test for extrema).

Let $f$ be continuous on $I = [a, b]$ and $c$ be an interior point of $I$. Let $f$ be differentiable on $(a, c)$ and $(c, b)$.

1. If $\exists$ a neighbourhood $(c-\delta, c+\delta) \subset I$ of $c$ such that $f'(x) \geq 0$ for $x \in (c-\delta, c)$ and $f'(x) \leq 0$ for $x \in (c, c+\delta)$, then $f$ has a local maximum at $c$.

2. If $\exists$ a neighbourhood $(c-\delta, c+\delta) \subset I$ of $c$ such that $f'(x) \leq 0$ for $x \in (c-\delta, c)$ and $f'(x) \geq 0$ for $x \in (c, c+\delta)$, then $f$ has a local minimum at $c$.

3. If $f'(x)$ keeps the same sign on ~~$(c-\delta, c+\delta)$~~ $(c-\delta, c)$ and $(c, c+\delta)$ then $f$ has no extremum at $c$.

Proof: 1. Let $x \in (c-\delta, c)$. Applying Mean Value Theorem to the function $f$ on $[x, c]$, we have

$$f(c) - f(x) = (c-x) f'(\xi) \quad \text{for some } \xi \in (x, c)$$

Since $f'(\xi) \geq 0$, we have $f(x) \leq f(c)$ for $x \in (c-\delta, c)$

Let $x \in (c, c+\delta)$. Applying mean value theorem to the function $f$ on $[c, x]$, we have $f(x) - f(c) = (x-c) f'(\eta)$ for some $\eta \in (c, x)$. Since $f'(\eta) \leq 0$ we have

$f(x) \leq f(c)$ for $x \in (c, c+\delta)$. It follows that

$f(c) \geq f(x)$ for all $x \in N(c,\delta) \cap I$, $N(c,\delta) = (c-\delta, c+\delta)$

So, $f$ has a local maximum at $c$

2. Similar proof as 1.

3. Let $f'(x) > 0$ for $x \in (c-\delta, c)$ and for $x \in (c, c+\delta)$.

Then $f(x) < f(c)$ for $x \in (c-\delta, c)$ and $f(c) < f(x)$ for $x \in (c, c+\delta)$. So $f$ has neither a maximum or a minimum at $c$. Similar proof if $f'(x) < 0$ for $x \in (c-\delta, c)$ and for $x \in (c, c+\delta)$.

Note: The converse of the theorem is not true

For example, let $f(x) = 2x^2 + x^2 \sin\frac{1}{x}$, $x \neq 0$

$= 0$, $x = 0$

Then $f$ has a local minimum at $0$.

$f'(x) = 4x + 2x \sin\frac{1}{x} - \cos\frac{1}{x}$, $x \neq 0$

$= 0$, $x = 0$

$f'$ takes both positive and negative values on both sides of $0$ (in the immediate neighbourhood).

Examples

1. Let $f(x) = |x|$, $x \in \mathbb{R}$

$f$ is continuous on $\mathbb{R}$, $f$ is not differentiable at $0$.

$f'(x) < 0$ for $x \in (-\delta, 0)$ and $f'(x) > 0$ for $x \in (0, \delta)$ for some $\delta > 0$. So, $f$ has a local minimum at $0$.

2. Let $f(x) = |x-1| + |x-2|$, $x \in [0,3]$.

Then $f(x) = 3 - 2x$, if $0 \leq x < 1$

$= 1$, if $1 \leq x \leq 2$

$= 2x - 3$, if $2 < x \leq 3$

$f$ is continuous on $[0,3]$. $f$ is not differentiable at 1 and 2. $f'(x) < 0$ for $x \in (1-\delta, 1)$, $f'(x) = 0$ for $x \in (1, 1+\delta)$ for ~~S~~ $\delta$ satisfying $0 < \delta < 1$. So, $f$ has a local minimum at 1.

$f'(x) = 0$ for $x \in (2-\delta, 2)$, $f'(x) > 0$ for $x \in (2, 2+\delta)$ for some $\delta$ satisfying $0 < \delta < 1$. So, $f$ has a local minimum at 2.

3. $f(x) = (x-1)^2 (x-3)^3$, $x \in \mathbb{R}$

$$f'(x) = 2(x-1)(x-3)^3 + 3(x-1)^2(x-3)^2$$

$$= (x-1)(x-3)^2(5x-9), \quad x \in \mathbb{R}$$

$f$ is continuous on $\mathbb{R}$, $f'(x) = 0$ at the points $1, 3, 9/5$

$f'(x) > 0$ for $x \in (1-\delta, 1)$ and $f'(x) < 0$ for $x \in (1, 1+\delta)$ for some $\delta > 0$. So, $f$ has a local maximum at 1

$f'(x) > 0$ for $x \in (3-\delta, 3)$ and $f'(x) > 0$ for $x \in (3, 3+\delta)$ for some $\delta > 0$. So, $f$ has neither a maximum nor a minimum at 3.

$f'(x) < 0$ for $x \in (9/5 - \delta, 9/5)$ and $f'(x) > 0$ for $x \in (9/5, 9/5 + \delta)$ for some $\delta > 0$. So, $f$ has a local minimum at $9/5$.

Now we state without proof a sufficient condition for the ~~ext~~ existence of maximum or minimum which involves higher order derivatives

Theorem 2.3.3 (Higher order derivative test for extrema)

Let $f: I \to \mathbb{R}$ and $c$ be an interior point of $I$.

If $f'(c) = f''(c) = \dots = f^{n-1}(c) = 0$ and $f^n(c) \neq 0$, then $f$ has

(i) no extremum at $c$ if $n$ be odd, and

(ii) a local extremum at $c$ if $n$ be even:

a local maximum if $f^n(c) < 0$, a local minimum if $f^n(c) > 0$

Worked examples 1. $f(x) = x^5 - 5x^4 + 5x^3 + 10$, $x \in \mathbb{R}$

Show that $f$ has a maximum at 1 and a minimum at 3 and $f$ has neither a maximum nor a minimum at 0.

Solution: For an extremum $f'(x) = 0$. $f'(x) = 0$ at $x = 1, 3, 0$

$f''(x) = 20x^3 - 60x^2 + 30x$. So, $f''(1) < 0$, $f''(3) > 0$, $f''(0) = 0$

Since $f'(1) = 0$ and $f''(1) < 0$, $f$ has a local maximum at 1

Since $f'(3) = 0$ and $f''(3) > 0$, $f$ has a local minimum at 3

Since $f'(0) = 0$, and $f''(0) = 0$, in order to decide the nature, we are to examine derivatives of higher order at 0.

$f'''(x) = 60x^2 - 120x + 30$. $f'''(0) = 30 \neq 0$

So, $f$ has neither a maximum nor a minimum at 0.

2. If $f'(x) = (x-a)^{2n}(x-b)^{2m+1}$ where $m, n$ are positive integers, show that $f$ has neither a maximum nor a minimum at $a$ and $f$ has a minimum at $b$.

Solution: $a$ is a multiple root of order $2n$ of the polynomial $f'(x)$. So, $a$ is a multiple root of order $2n-1$ of the polynomial $f''(x)$, a multiple root of order $2n-2$ of the polynomial $f'''(x)$, ...,