a simple root of the polynomial $f^{2n}(x)$. Also a is not a root of $f^{2n+1}(x)$.

So, $f'(a) = f''(a) = \cdots = f^{2n}(a) = 0$ and $f^{2n+1}(a) \neq 0$

Since $2n+1$ is odd, f has neither a maximum nor a minimum at a.

Let h be arbitrarily small positive number.

$$f'(b-h) = (b-h-a)^{2n}(-h)^{2m+1} < 0$$

$$f'(b+h) = (b+h-a)^{2n} h^{2m+1} > 0$$

f is continuous at b. $f'(x) < 0$ for $x \in (b-\delta, b)$ and $f'(x) > 0$ for $x \in (b, b+\delta)$ for some $\delta > 0$.

Hence f has a local minimum at b.

3. Find the local extremum points of the function

$$f(x) = \frac{x^2}{(1-x)^3}, \quad x \neq 1$$

Solution: $$f'(x) = \frac{2(1-x)^3 x + 3x^2(1-x)^2}{(1-x)^6} = \frac{x(1-x)^2(x+2)}{(1-x)^6}$$

$$= \frac{x(x+2)}{(1-x)^4}.$$

So, $f'(x) = 0$ at $x = -2, 0$

Let h be an arbitrarily small positive number.

$$f'(-2-h) > 0, \quad f'(-2) = 0, \quad f'(-2+h) < 0$$

$$f'(0-h) < 0, \quad f'(0) = 0, \quad f'(0+h) > 0$$

f is continuous at -2, $f'(x) > 0$ for $x \in (-2-\delta, -2)$ and $f'(x) < 0$ for $x \in (-2, -2+\delta)$ for some $\delta > 0$

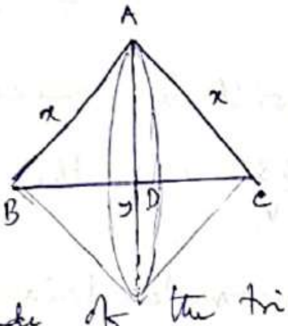
f is continuous at 0, $f'(x) < 0$ for $x \in (-\delta, 0)$ and $f'(x) > 0$ for $x \in (0, \delta)$ for some $\delta > 0$

Hence f has a local maximum at -2 and a local minimum at 0.

Application of principle of maximum and/minimum in some geometric problems

1. The perimeter of an isosceles triangle is $2s$. What must its sides be so that the volume of the solid generated by revolving the triangle about the base is the greatest possible?

Solution Let x be the length of the equal sides and y be the length of the base of the isosceles triangle



Here $AB = AC = x$

and $BC = y$

Given that $2x + y = 2s$

AD is the altitude of the triangle. So,

$$AD = \sqrt{x^2 - \left(\frac{y}{2}\right)^2} = \frac{\sqrt{4x^2 - y^2}}{2}$$

Let V be the volume of the solid generated when the ~~triang~~ triangle is revolved about the base.

$$\text{Then } V = 2 \cdot \frac{1}{3} \pi (AD)^2 \cdot \frac{y}{2}$$

$$= \frac{2}{3} \pi \frac{(4x^2 - y^2)}{4} \cdot \frac{y}{2}$$

$$= -\frac{\pi}{12} \left(4x^2 - (2x - 2s)^2\right)(2x - 2s)$$

$$= -\frac{\pi}{12} \left(4x^2 - (4x^2 - 8xs + 4s^2)\right)(2x - 2s)$$

$$= -\frac{\pi}{12} \left(4x^2 - 4x^2 + 8xs - 4s^2\right)(2x - 2s)$$

$$= -\frac{2\pi}{3} \left(2xs - s^2\right)(x - s)$$

$$\frac{dv}{dx} = -\frac{2\pi}{3}\left[2xs - s^2 + (x-s)2s\right]$$

$$= -\frac{2\pi}{3}\left[4xs - 3s^2\right]$$

For maximum value of V, $\frac{dV}{dx} = 0$

So, $4sx = 3s^2$ or, $x = \frac{3s^2}{4s} = \frac{3s}{4}$

Now $\frac{d^2V}{dx^2} = -\frac{2\pi}{3}\left[4s\right] = -\frac{8\pi s}{3} < 0$

So, V is maximum when $x = \frac{3s}{4}$

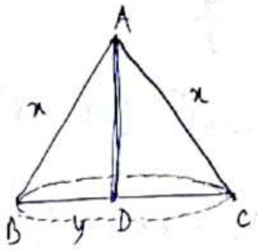
As $2x+y = 2s$, so, $y = 2s - \frac{3s}{2}$

$= \frac{4s-3s}{2} = \frac{s}{2}$

So, volume generated will be maximum when the equal sides are $\frac{3s}{4}, \frac{3s}{4}$ and the base is $\frac{s}{2}$

5. The perimeter of an isosceles triangle is $2s$. What must its sides be so that the volume of the solid generated by revolving the triangle about the altitude upon the base is the greatest possible

Solution: Let ΔABC be the isosceles triangle where $AB = AC = x$ and $BC = y$



AD = altitude $= \sqrt{x^2 - \left(\frac{y}{2}\right)^2}$

$= \frac{\sqrt{4x^2 - y^2}}{2}$

Given also $2x + y = 2s$ ...(i)

Let V be the volume of the solid generated by revolving the triangle about the altitude upon the base.

Then $V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 \cdot AD$

$$= \frac{1}{3}\pi \frac{y^2}{4} \frac{\sqrt{4x^2-y^2}}{2}$$

$$= \frac{\pi}{24} y^2 \sqrt{(2s-y)^2-y^2} \quad [\text{from (i)}]$$

$$= \frac{\pi}{24} y^2 \sqrt{4s^2-4sy+y^2-y^2}$$

$$= \frac{\pi}{24} y^2 \sqrt{4s^2-4sy}$$

$$\frac{dV}{dy} = \frac{\pi}{24}\left[y^2 \frac{(-4s)}{2\sqrt{4s^2-4sy}} + 2y\sqrt{4s^2-4sy}\right]$$

$$= \frac{\pi}{24}\left[2y\sqrt{4s^2-4sy} - \frac{2sy^2}{\sqrt{4s^2-4sy}}\right]$$

$$= \frac{\pi}{12}\left[y\sqrt{4s^2-4sy} - \frac{sy^2}{\sqrt{4s^2-4sy}}\right]$$

When V is maximum,

$$\frac{dV}{dx} = 0, \text{ So, } y\sqrt{4s^2-4sy} = \frac{sy^2}{\sqrt{4s^2-4sy}}$$

or, $sy^2 = y(4s^2-4sy)$

or, $sy^2 = 4s^2y - 4sy^2$

or, $5sy^2 = 4s^2y$

or, $y = \frac{4s}{5}$ (As $y \neq 0$)

Now $\cancel{\frac{d^2V}{dy^2} = \frac{\pi}{12}\left[\frac{y(-4s)}{2\sqrt{4s^2-4sy}} - \frac{\sqrt{4s^2-4sy}}{4s^2-4sy}\right.}$