Now, $\dfrac{d^2V}{dy^2} = \dfrac{\pi}{12}\left[\dfrac{y(-4s)}{2\sqrt{4s^2-4sy}} + \sqrt{4s^2-4sy} - \dfrac{\sqrt{4s^2-4sy}\,(2sy) - sy^2\left(\frac{1}{2\sqrt{4s^2-4sy}}\right)(-4s)}{4s^2-4sy}\right]$

$$= \frac{\pi}{12}\left[\frac{-2sy}{\sqrt{4s^2-4sy}} + \sqrt{4s^2-4sy} - \frac{2sy\sqrt{4s^2-4sy} + \frac{2s^2y^2}{\sqrt{4s^2-4sy}}}{4s^2-4sy}\right]$$

$$= \frac{\pi}{12}\left[\frac{-4sy}{\sqrt{4s^2-4sy}} + \sqrt{4s^2-4sy} - \frac{2s^2y^2}{(4s^2-4sy)^{3/2}}\right]$$

When $y = \dfrac{4s}{5}$

$$\frac{d^2V}{dy^2} = \frac{\pi}{12}\left[\frac{-\frac{16s^2}{5}}{\sqrt{4s^2-\frac{16s^2}{5}}} + \sqrt{4s^2-\frac{16s^2}{5}} - \frac{2s^2\times\frac{16s^2}{25}}{\left(4s^2-\frac{16s^2}{5}\right)^{3/2}}\right]$$

$$= \frac{\pi}{12}\left[\frac{-\frac{16s^2}{5}}{\frac{2s}{\sqrt5}} + \frac{2s}{\sqrt5} - \frac{\frac{32s^4}{25}}{\frac{8s^3}{5^{3/2}}}\right]$$

$$= \frac{\pi}{12}\left[-\frac{16s^2}{5}\times\frac{\sqrt5}{2s} + \frac{2s}{\sqrt5} - \frac{32s^4}{25}\times\frac{5^{3/2}}{8s^3}\right]$$

$$= \frac{\pi}{12}\left[-\frac{8}{\sqrt5}s + \frac{2s}{\sqrt5} - \frac{4s}{\sqrt5}\right]$$

$$= \frac{\pi}{12}\left(-\frac{10s}{\sqrt5}\right) < 0$$

So the volume is maximum when

$y = \dfrac{4s}{5}$ and $x = \dfrac{2s - \frac{4s}{5}}{2} = \dfrac{3s}{5}$

So, the sides ~~of~~ of the triangle should be $\dfrac{3s}{5}$, $\dfrac{3s}{5}$ and $\dfrac{4s}{5}$.

—

6. Show that the maximum rectangle inscribed in a circle is a square.

Solution: Let the equation of the circle be $x^2+y^2=a^2$. Then the co-ordinates of any point P on the circle is $(a\cos\theta, a\sin\theta)$.

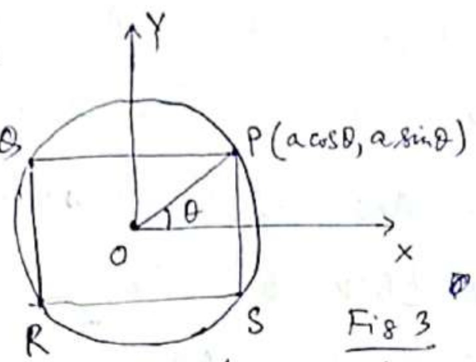

Fig 3

Let PQRS be the rectangle inscribed in the circle.

So, $PQ = 2a\cos\theta$, $QR = 2a\sin\theta$

So, the area A of the rectangle PQRS is

$A = 4a^2\sin\theta\cos\theta$

For extremum value of A, $\frac{dA}{d\theta} = 0$ gives

$\cos 2\theta = 0$ or, $\theta = \pi/4$

$\frac{d^2A}{d\theta^2} = -8a^2\sin 2\theta = -8a^2 < 0$ for $\theta = \pi/4$

So, $\theta = \pi/4$ gives the maximum value of A and maximum value of A occurs when $PQ = PS = \frac{2a}{\sqrt{2}} = \sqrt{2}\,a$

So, PQRS is a square.

7. Show that the triangle of maximum area that can be inscribed in a circle is an equilateral one.

Solution: Let ABC be a triangle inscribed in the circle with centre O and radius a.

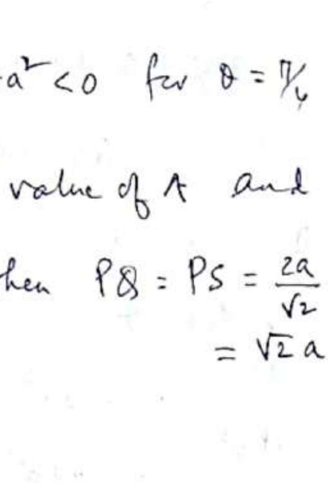

Fig. 4

Of all triangles, with AB as base, the area is maximum when the perpendicular distance of C from AB is maximum.

So, for the triangle ABC with maximum area, C must lie on the diameter perpendicular to AB.

Then ABC is an isosceles triangle.

Let $\angle BCD = \theta$, then $\angle BOD = 2\theta$, D being the mid point of AB.

$\therefore$ $CD = a + a\cos 2\theta$ and $AB = 2a\sin 2\theta$

$\therefore$ Area of the triangle ABC is $S = a\sin 2\theta\,(a + a\cos 2\theta)$

For extremum value of S, $\frac{dS}{d\theta} = 0$ which gives

$$2a\cos 2\theta(a + a\cos 2\theta) + a\sin 2\theta\,(-2a\sin 2\theta) = 0$$

or, $2a^2(\cos^2 2\theta - \sin^2 2\theta) + 2a^2\cos 2\theta = 0$

or, $2a^2(\cos 4\theta + \cos 2\theta) = 0$

which $\cos 3\theta \cos\theta = 0$. So, either $\theta = \frac{\pi}{6}$ or $\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{2}$, is not possible, so, $\theta = \frac{\pi}{6}$, and for this value θ

$$\frac{d^2S}{d\theta^2} = -2a^2(4\sin 4\theta + 2\sin 2\theta)$$
$$= -2a^2\left(4\sin\frac{2\pi}{3} + 2\sin\frac{\pi}{3}\right) < 0$$

$\therefore$ $\theta = \frac{\pi}{6}$ gives maximum value of S and then

$$\angle ACB = \frac{\pi}{3} = \angle CAB = \angle ABC$$

So, the triangle is equilateral

8. Show that the semi-vertical angle of the cone of maximum volume and given slant height is $\tan^{-1}\sqrt{2}$,

Solution: Let α be the semi-vertical angle of the cone whose slant height is l.

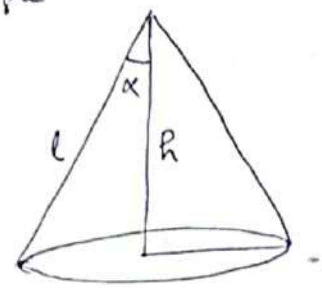

Fig. 5

The radius r of the base is $r = l\sin\alpha$

and height h is given by

$h = l\cos\alpha$

Volume $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi l^3 \sin^2\alpha \cos\alpha$

For extremum value of V, $\frac{dV}{d\alpha} = 0$ gives

$2\sin\alpha\cos^2\alpha - \sin^3\alpha = 0$

or, $\tan\alpha = \pm\sqrt{2}$

For $\tan\alpha = \sqrt{2}$,

$$\frac{d^2V}{d\alpha^2} = \frac{1}{3}\pi l^3 \left(2\cos^3\alpha - 7\sin^2\alpha\cos\alpha\right) < 0$$

So, V is maximum when $\tan\alpha = \sqrt{2}$

or, $\alpha = \tan^{-1}\sqrt{2}$.

9. Find the point on the parabola $2y = x^2$ which is nearest to the point $(0,3)$.

Solution: Let $P(x,y)$ be the point on the parabola which is nearest to the point $A(0,3)$.

$AP = \sqrt{x^2 + (y-3)^2} = \sqrt{2y + y^2 - 6y + 9}$

$= \sqrt{y^2 - 4y + 9}$ $\therefore AP^2 = y^2 - 4y + 9 = f(y)$ (say)

$f'(y) = 0$ for minimum value of AP, or $y = 2$

$f''(y) = 2 > 0$. So $y = 2$ gives $x^2 = 4$ or $x = \pm 2$. So, we get the points $(\pm 2, 2)$ which are nearest to $(0,3)$ and the distance is $\sqrt{5}$.

END OF MY PORTION OF NOTES