Hence f is differentiable at 3 and $f'(3) = -5$

So, the ~~derivat~~ derived function f' is defined by

$$f'(x) = 1, \quad 0 \leq x < 1$$
$$= -2x, \quad 1 < x < 2$$
$$= 1 - 2x, \quad 2 < x \leq 3$$

The domain of f' is $[0,1) \cup (1,2) \cup (2,3]$

Theorem 1.1.4 Let I be an interval and $c \in I$. Let the functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be differentiable at c. Then

(i) $f+g$ is differentiable at c and $(f+g)'(c) = f'(c) + g'(c)$

(ii) if $k \in \mathbb{R}$, kf is differentiable at c and $(kf)'(c) = kf'(c)$

(iii) $f.g$ is differentiable at c and $(f.g)'(c) = f'(c)g(c) + f(c)g'(c)$

(iv) if $g(c) \neq 0$, f/g is differentiable at c and $(f/g)'(c)$ is given by

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$$

Proof : Proofs of (i) and (ii) are left as exercises

(iii) Let $h = f.g$. Then for $x \in I$, $x \neq c$,

$$\frac{h(x) - h(c)}{x - c} = \frac{f(x)g(x) - f(c)g(c)}{x - c}$$

$$= \frac{f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{x - c}$$

$$= \frac{f(x) - f(c)}{x - c} \cdot g(x) + f(c) \cdot \frac{g(x) - g(c)}{x - c}$$

Since g is differentiable at c and hence g is continuous at c by Theorem 1.1.3. So, $\lim_{x \to c} g(x) = g(c)$. Since f is differentiable at c, $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$. Since g is differentiable at c, so, $\lim_{x \to c} \frac{g(x) - g(c)}{x - c} = g'(c)$. So, we have

$$\lim_{x \to c} \frac{h(x) - h(c)}{x - c} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} g(x) + f(c) \cdot \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

$= f'(c)\,g(c) + f(c)\,g'(c)$

So, $f\cdot g$ is differentiable at c and $(f\cdot g)'(c) = f'(c)\,g(c) + f(c)\,g'(c)$.

(iv) Let $h = f/g$. Since g is differentiable at c, g is continuous at c. Since $g(c) \neq 0$, there exists a neighbourhood $N(c)$ of c such that $g(x) \neq 0$ for all $x \in N(c) \cap I$. So, for $x \in N(c) \cap I$, $x \neq c$,

$$\frac{h(x)-h(c)}{x-c} = \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x-c} = \frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)(x-c)}$$

$$= \frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)}{g(x)g(c)(x-c)}$$

$$= \frac{1}{g(x)g(c)}\left[\frac{f(x)-f(c)}{x-c}\cdot g(c) - f(c)\cdot\frac{g(x)-g(c)}{x-c}\right]$$

Since g is continuous at c, $\lim_{x\to c} g(x) = g(c)$

Since f and g are differentiable at c,

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = f'(c) \text{ and } \lim_{x\to c}\frac{g(x)-g(c)}{x-c} = g'(c)$$

$$\text{So, } \lim_{x\to 0}\frac{h(x)-h(c)}{x-c} = \frac{1}{g(c)\cdot g(c)}\left[f'(c)g(c) - f(c)g'(c)\right] = \frac{1}{(g(c))^2}\left[f'(c)g(c) - f(c)g'(c)\right]$$

So, ~~(f/g) is differe~~ f/g is differentiable at c

and $$\left(f/g\right)'(c) = \frac{f'(c)\,g(c) - f(c)\,g'(c)}{(g(c))^2}.$$

As an immediate consequence of Theorem 1.1.4, we have the following theorem:

Theorem 1.1.5 Let I be an interval and let the functions $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ be both differentiable on a subset $D \subset I$. Then

(i) $f+g$ is differentiable on D and $(f+g)'(x) = f'(x) + g'(x)$, $x \in D$

(ii) if $k \in \mathbb{R}$, kf is differentiable ~~at~~ on D and $(kf)'(x) = k f'(x)$, $x \in D$

(iii) $f\cdot g$ is differentiable on D and $(f\cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$, $x \in D$

(iv) if $g'(x) \neq 0$ on D, f/g is differentiable on D and $(f/g)'$ is given by

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}, \quad x \in D$$

Theorem 1.1.6 Let I and J be intervals. Let $f: I \to \mathbb{R}$ and ~~[illegible]~~ $g: J \to \mathbb{R}$ be two functions such that $f(I) \subset J$. Let $c \in I$. Let ~~[illegible]~~ f be differentiable at c and g be differentiable at $f(c)$. Then the composite function $g \circ f$ (defined by $(g\circ f)(x) = g(f(x))$) ~~is differentiable~~ is differentiable at c and $(g\circ f)'(c) = g'(f(c)) \cdot f'(c)$

Proof: Let $f(c) = d$. Since g is differentiable at d

$\lim_{y \to d} \frac{g(y) - g(d)}{y - d} = g'(d)$. Since f is differentiable at c,

$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$. Let us define a function $G: J \to \mathbb{R}$ by

$$G(y) = \frac{g(y) - g(d)}{y - d}, \quad y \neq d$$
$$= g'(d), \quad y = d$$

Then $\lim_{y \to d} G(y) = \lim_{y \to d} \frac{g(y) - g(d)}{y - d} = g'(d)$ since g is differentiable at d.

$= G(d)$, by definition. This shows that G is continuous at d.

Since f is continuous at c and G is continuous at $d (= f(c))$, the composite function $G\circ f$ is continuous at c. Hence $\lim_{x \to c} (G\circ f)(x) = (G\circ f)(c)$

But $\lim_{x \to c} (G\circ f)(x) = \lim_{x \to c} \frac{g(f(x)) - g(d)}{f(x) - d}$, by definition of G.

$$= \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)}$$

Therefore, $\lim_{x \to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} = g'(d)$, since $(G\circ f)(c) = G(f(c))$
$= G(d) = g'(d)$

We also have $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$

Hence $\lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} = \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \lim_{x \to 0} \frac{f(x) - f(c)}{x - c} = g'(d) \cdot f'(c)$

So, the function $g \circ f$ is differentiable at c and
$(g \circ f)'(c) = g'(d) f'(c) = g'(f(c)) f'(c)$. This completes the proof.

As an immediate consequence of the theorem 1.1.6, we have the following theorem:

Theorem 1.1.7. Let I and J be intervals and let $f: I \to \mathbb{R}$, $g: J \to \mathbb{R}$ be two functions such that $f(I) \subset J$. If f is differentiable on I and g is differentiable on $f(I)$, then the composite function $g \circ f$ (defined by $(g \circ f)(x) = g(f(x))$) is differentiable on I and $(g \circ f)'(x) = g'(f(x)) f'(x)$, $x \in I$.

Example (continued)

5. Find the derived function of f where $f(x) = x^{\alpha}$, $x > 0$ and $\alpha \in \mathbb{R}$.

Solution: Let $g(x) = \alpha \log x$, $x > 0$ and $h(x) = e^{x}$, $x \in \mathbb{R}$

Then $f(x) = (h \circ g)(x)$, $x > 0$ and $f'(x) = h'(g(x)) \cdot g'(x)$, $x > 0$

But $h'(x) = e^{x}$ and hence $h'(g(x)) = e^{\alpha \log x} = x^{\alpha}$ and $g'(x) = \frac{\alpha}{x}$.

So, $f'(x) = x^{\alpha} \cdot \frac{\alpha}{x} = \alpha x^{\alpha - 1}$

Theorem 1.1.8 Let $I \subset \mathbb{R}$ be an interval and let $f: I \to \mathbb{R}$ be a strictly monotone ~~and~~ and continuous function on I. Let $J = f(I)$ and let $g: I \to \mathbb{R}$ be the inverse to f. If f is differentiable at $c \in I$ and $f'(c) \neq 0$ then g is differentiable at $d (= f(c))$ and $g'(d) = \frac{1}{f'(c)}$

Proof: Let $y \in J$, $y \neq d$. Let $g(y) = x \in I$. Then $f(x) = y$ and since f is strictly monotone on I, $x \neq c$

$$\frac{g(y) - g(d)}{y - d} = \frac{g(f(x)) - g(f(c))}{f(x) - f(c)}$$

Since f is strictly monotone and continuous on I, g is