continuous on J. As $y \to d$, $g(y) \to g(d)$

Since $(g \circ f)(x) = x$ for all $x \in I$, it follows that $x \to c$ as $y \to d$

Therefore, $\lim\limits_{y \to d} \frac{g(y) - g(d)}{y - d} = \lim\limits_{x \to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} = \lim\limits_{x \to c} \frac{(g \circ f)(x) - (g \circ f)(c)}{f(x) - f(c)}$

$= \lim\limits_{x \to c} \frac{x - c}{f(x) - f(c)}$. Since f is differentiable at c,

$\lim\limits_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$ and since $f'(c) \neq 0$,

$\lim\limits_{x \to c} \frac{x - c}{f(x) - f(c)} = \frac{1}{f'(c)}$

Therefore, $\lim\limits_{y \to d} \frac{g(y) - g(d)}{y - d} = \frac{1}{f'(c)}$. That is, $g'(d) = \frac{1}{f'(c)}$

Examples (continued)

6. Let $f(x) = x^2$, $x \in [0, \infty)$. f is strictly increasing and continuous on $[0, \infty)$. Let $I = [0, \infty)$. Then $f(I) = [0, \infty)$. The inverse function g defined by $g(y) = \sqrt{y}$, $y \in [0, \infty)$ is continuous on $[0, \infty)$.
f is differentiable on $[0, \infty)$ and $f'(x) = 2x$, $x \in [0, \infty)$
$f'(x) \neq 0$ on $(0, \infty)$. Let $I_1 = (0, \infty)$. Then $f(I_1) = (0, \infty)$
Hence $g'(y)$ exists for all $y \in (0, \infty)$ and

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{2x} = \frac{1}{2g(y)} = \frac{1}{2\sqrt{y}}, \; y \in (0, \infty)$$

7. Let $f(x) = e^x$, $x \in \mathbb{R}$. f is strictly increasing and continuous on $\mathbb{R}$. $f(\mathbb{R}) = (0, \infty)$. The inverse function g of f is defined by $g(y) = \log y$, $y \in (0, \infty)$ and it is continuous on $(0, \infty)$. f is differentiable on $\mathbb{R}$ and $f'(x) = e^x \neq 0$ on $\mathbb{R}$. Hence $g'(y)$ exists for all $y \in (0, \infty)$ and $g'(y) = \frac{1}{f'(x)} = \frac{1}{e^x} = \frac{1}{e^{\log y}} = \frac{1}{y}$, $y \in (0, \infty)$

$x \in (0, \pi/2)$ and ~~sect~~ $\sec x \tan x = -y\sqrt{y^2-1}$, $x \in (\pi/2, \pi)$.

So, $g'(y) = \dfrac{1}{|y|\sqrt{y^2-1}}$, $y \in (-\infty, -1) \cup (1, \infty)$

Worked Examples

1. A function f is defined on some neighbourhood of c and f is differentiable at c. Prove that $\lim\limits_{h \to 0} \dfrac{f(c+h) - f(c-h)}{2h} = f'(c)$

Show by example that the limit may exist even if $f'(c)$ does not exist.

Solution: $\lim\limits_{h \to 0+} \dfrac{f(c+h) - f(c-h)}{2h} = \lim\limits_{h \to 0+} \left[\dfrac{f(c+h) - f(c)}{2h} + \dfrac{f(c-h) - f(c)}{-2h} \right]$

$$= \lim_{h \to 0+} \frac{f(c+h) - f(c)}{2h} + \lim_{k \to 0-} \frac{f(c+k) - f(c)}{2k}$$

$$= \frac{1}{2} Rf'(c) + \frac{1}{2} Lf'(c) = \frac{1}{2} f'(c) + \frac{1}{2} f'(c) = f'(c)$$

as $f'(c)$ exists

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So, $\lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{2h} = f'(c)$

Let $f(x) = |x|$ and $c = 0$

Then $\lim\limits_{h \to 0} \dfrac{f(c+h) - f(c-h)}{2h} = \lim\limits_{h \to 0} \dfrac{f(h) - f(-h)}{2h}$

$$= \lim_{h \to 0} \frac{|h| - |-h|}{2h}$$

Let $h \to 0+$ then $\lim\limits_{h \to 0+} \dfrac{|h| - |-h|}{2h} = \lim\limits_{h \to 0+} \dfrac{h - h}{2h} = 0$

$x \in (0, \pi/2)$ and ~~sec~~ $\sec x \tan x = -y\sqrt{y^2-1}$, $x \in (\pi/2, \pi)$.

So, $g'(y) = \dfrac{1}{|y|\sqrt{y^2-1}}$, $y \in (-\infty, -1) \cup (1, \infty)$

Worked Examples

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So, $\lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{2h} = f'(c)$

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Let $h \to 0+$ then $\lim\limits_{h \to 0+} \dfrac{|h| - |-h|}{2h} = \lim\limits_{h \to 0+} \dfrac{h - h}{2h} = 0$

Let $h \to 0-$ Then $\lim_{h\to 0-} \frac{|h|-|-h|}{2h} = \lim_{h\to 0-} \frac{-h-(-h)}{2h} = 0$

So, $\lim_{h\to 0} \frac{|h|-|-h|}{h} = 0$ So, at $c=0$; $\lim \frac{f(c+h)-f(c-h)}{2h} = 0$

Now $\underset{h\to 0+}{Lt} \frac{f(h)-f(0)}{h} = \underset{h\to 0+}{Lt} \frac{|h|}{h} = \underset{h\to 0+}{Lt} \frac{h}{h} = 1$

and $\underset{h\to 0-}{Lt} \frac{f(h)-f(0)}{h} = \underset{h\to 0-}{Lt} \frac{|h|}{h} = \underset{h\to 0-}{Lt} \frac{-h}{h} = -1$

So, $f'(0)$ does not exist.

2. A function f is defined on $\mathbb{R}$ by $f(x) = x^2 \sin\frac{1}{x}$, $x \neq 0$
$= 0$, $x = 0$

Show that f is differentiable at 0 but f' is not continuous at 0.

Solution: $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} x\sin\frac{1}{x} = 0$ since $\lim_{x\to 0} x = 0$ and $\sin\frac{1}{x}$ is bounded on some deleted neighbourhood of 0. Hence $f'(0) = 0$. When $x \neq 0$, $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$.

Thus the derived function f' is defined by

$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}, \quad x \neq 0$$
$$= 0, \quad x = 0$$

$\lim_{x\to 0} f'(x)$, since $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

So, f' is not continuous at 0.

3. A function $f: \mathbb{R} \to \mathbb{R}$ is defined by $f(0) = 0$ and

$f(x) = 0$, if x is irrational
$= \frac{1}{q}$, if $x = p/q$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\gcd(p,q) = 1$

Show that f is not differentiable at 0.