Solution: $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x}$

Let $\phi(x) = \frac{f(x)}{x}$. Let $\{x_n\}$ be a sequence of rational points converging to zero, where $x_n = \frac{1}{n}$, $n \in \mathbb{N}$

Then $\lim_{n\to\infty} \phi(x_n) = \lim_{n\to\infty} \frac{f(x_n)}{x_n} = \lim_{n\to\infty} n \cdot \frac{1}{n} = 1$

Let $\{y_n\}$ be a sequence of irrational points converging to 0. Let $y_n = \frac{\sqrt{2}}{n}$. Then $\lim_{n\to\infty} \phi(y_n) = \lim_{n\to\infty} \frac{f(y_n)}{y_n} = 0$ since $f(y_n) = 0$ for all $n \in \mathbb{N}$. So, $\lim_{x\to 0} \phi(x)$ does not exists, since for two sequences $\{x_n\}$ and $\{y_n\}$ both converging to zero, the sequence $\{\phi(x_n)\}$ and $\{\phi(y_n)\}$ converges to two different limits

Hence f is not differentiable at 0.

Theorem 2.1.1 (Darboux Theorem) Let $I = [a,b]$ and a function $f: I \to \mathbb{R}$ be differentiable on I. Let $f'(a) \neq f'(b)$. If k be a real number lying between $f'(a)$ and $f'(b)$ then there exists a point c in (a,b) such that

$f'(c) = k$

Proof: Without loss of generality, let $f'(a) < k < f'(b)$

Let us define $g: [a,b] \to \mathbb{R}$ by $g(x) = f(x) - kx$, $x \in [a,b]$

g is differentiable on $[a,b]$ and ~~so~~ therefore g is continuous on $[a,b]$. Consequently, g will attain the greatest lower bound at some point c in $[a,b]$.

$g'(a) = f'(a) - k < 0$. This implies g is decreasing at a.

So, there exists a positive δ such that $g(x) < g(a)$

for all $x \in [a, b]$ satisfying $a < x < a+\delta$. This shows that $g(a)$ is not the greatest lower bound.

$g'(b) = f'(b) - k > 0$. This implies g is increasing at b.
Therefore, there exists a positive δ' such that
$g(x) < g(b)$ for all $x \in [a, b]$ satisfying $b - \delta' < x < b$
This shows that $g(b)$ is not the greatest lower bound in $[a, b]$. So, $c \neq a$, $c \neq b$. So, $a < c < b$

Since $c \in (a, b)$, $g'(c)$ exists, we prove that $g'(c) = 0$
Let $g'(c) > 0$. Then there exist a positive δ'' such that $g(x) < g(c)$ for $x \in [a, b]$ satisfying $c - \delta'' < x < c$
This contradicts that $g(c)$ is the ~~greast~~ greatest lower bound on $[a, b]$. So, $g'(c) \not> 0$. Similarly, we can prove that $g'(c) \not< 0$. So, $g'(c) = 0$, i.e. $f'(c) = k$ and the theorem is established.

Corollary 2.1.2 Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be differentiable on I. If $f'(a) f'(b) < 0$ then there exists a point c on (a, b) such that $f'(c) = 0$

Worked Example

1. Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = 0$, $x \in [-1, 0]$
$= 1$, $x \in (0, 1]$

Does there exist a function g such that $g'(x) = f(x)$, $x \in [-1, 1]$

Solution : If possible, let there ~~exists~~ be a function $g : [-1, 1] \to \mathbb{R}$ such that $g'(x) = f(x)$, $x \in [-1, 1]$

Then g is differentiable on $[-1, 1]$

and $g'(x) = 0$, $x \in [-1, 0]$
$= 1$, $x \in (0, 1]$

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Since g is differentiable on $[-1,1]$ and $g'(-1) \neq g'(1)$, by Darboux theorem g' must assume every value between $g'(-1)$ and $g'(1)$, i.e., between 0 and 1 on $[-1,1]$, a contradiction as $g'(x)$ assumes only two values 0 and 1 only. So g does not exist.

Theorem 2.1.2 (Rolle's theorem)

Let a function $f : [a,b] \to \mathbb{R}$ be such that

(i) f is continuous on $[a,b]$

(ii) f is differentiable on (a,b) and

(iii) $f(a) = f(b)$.

Then there exists at least one real number c in (a,b) such that $f'(c) = 0$

Proof: Since f is continuous in the closed interval $[a,b]$, it is bounded and attains its bounds.

Thus, if m and M are the infimum and supremum of f on $[a,b]$, then $\exists$ points c and d in $[a,b]$ such that $f(c) = m$, $f(d) = M$

Case 1 $m = M$

In this f is constant over $[a,b]$ and so

$$f'(x) = 0 \quad \forall x \in [a,b]$$

Case 2 $m \neq M$

As $m \neq M$, both of these can not be equal to the same quantity $f(a)$. At least one of these, say, m is different from $f(a)$ $(= f(b))$ so that

$$f(c) = m \neq f(a) \Rightarrow c \neq a$$

$$f(c) = m \neq f(b) (= f(a)) \Rightarrow c \neq b$$

So $c \in (a,b)$. We shall now show that

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c is the point where $f'(c) = 0$.

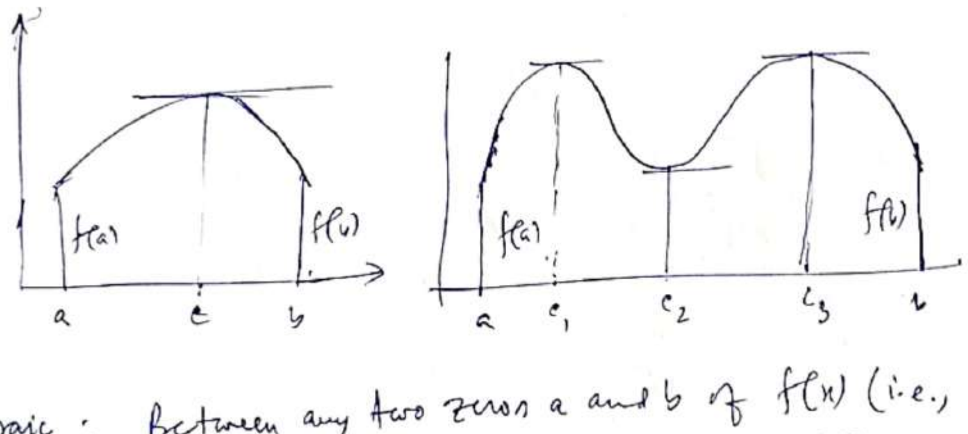
If $f'(c) < 0$ then $\exists$ an interval ~~Je~~ $(c, c+\delta_1)$, $\delta_1 > 0$ for every point x of which $f(x) < f(c) = m$ which contradicts the fact that m is the infimum.

If $f'(c) > 0$, $\exists$ an interval $(c-\delta_2, c)$, $\delta_2 > 0$ for every point x of which $f(x) < f(c) = m$, which is also a contradiction.

Hence the only possibility is $f'(c) = 0$.

Interpretation of Rolle's theorem

Geometric: Let the curve $y = f(x)$, which is continuous on $[a, b]$ and derivable on (a, b), be drawn. The Theorem simply states that between two points with equal ordinates on the graph of f, there exists at least one point where the tangent is parallel to x-axis.



Algebraic: Between any two zeros a and b of $f(x)$ (i.e., between any two roots a and b of $f(x) = 0$) there exists at least one zero of $f'(x)$.

Note: The conditions of Rolles' theorem is sufficient but may not be necessary. That mean there is function $f(x)$ such $f'(c) = 0$ for some $c \in (a, b)$ but f may not satisfy all the conditions of Rolle's Theorem.

Let f be defined on $[-1, 2]$ by