$f(x) = |x| + |x-1|,\ x \in [-1, 2]$.

$$f(x) = 2x-1, \quad 1 < x \le 2$$
$$= 1 \quad 0 \le x \le 1$$
$$= 1-2x \quad -1 \le x < 0$$

f is continuous on $[-1,2]$. f is not differentiable at 0 and 1; $f(-1) = f(2) = 3$. So, f does not satisfy the second ~~cont.~~ condition of Rolle's theorem. But $f'(x) = 0$ for all $x \in (0,1)$

<u>Theorem 2.1.3</u> (Mean Value Theorem of Lagrange)

Let a function $f : [a,b] \to \mathbb{R}$ be such that

(i) f is continuous on $[a,b]$ and

(ii) f is differentiable on (a,b)

then there exists at least one real number $c \in (a,b)$ such that
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proof: Let us consider a function $\phi(x) = f(x) + Ax,\ x \in [a,b]$, where A is a constant to be determined such that $\phi(a) = \phi(b)$

So, $A = -\dfrac{f(b)-f(a)}{b-a}$

Now ϕ, being the sum of two continuous and derivable functions, is itself

(i) continuous on $[a,b]$

(ii) differentiable on (a,b) and

(iii) $\phi(a) = \phi(b)$

Therefore, by Rolle's Theorem, $\exists$ a real number $c \in (a,b)$ such that $\phi'(c) = 0$.

but $\phi'(x) = f'(x) + A$

So, $\phi'(c) = f'(c) + A = 0$ or, $f'(c) = -A = \dfrac{f(b)-f(a)}{b-a}$

Another statement : If in the statement of theorem, b is replaced by $a+h$, then the number $c \in (a,b)$ may be written as $a+\theta h$, where $0<\theta<1$. Thus

$$f(a+h)-f(a) = h f'(a+\theta h)$$

$$\text{or,} \quad f(a+h) = f(a) + h f'(a+\theta h)$$

2.1.4 Deductions :

1. If a function $f:[a,b] \to \mathbb{R}$ satisfies the conditions of the Mean Value Theorem and $f'(x)=0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.

Proof: Let x_1, x_2 (where $x_1<x_2$) be any two distinct points of $[a,b]$. Hence by Lagrange's Mean Value Theorem, $\exists$ real number $c \in (x_1,x_2)$

such that $$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) = 0$$

So, $f(x_2) = f(x_1)$

Hence, the function keeps the same value and is therefore constant on $[a,b]$.

2. If two functions have equal derivatives at all points of (a,b), then they differ only by a constant

Proof : Exercise

3. If a function f is (i) continuous on $[a,b]$, (ii) derivable on (a,b), and (iii) $f'(x)>0$, $\forall x \in (a,b)$, then f is strictly increasing on $[a,b]$

Proof: Let x_1, x_2 (where $x_1<x_2$) be any two distinct points of $[a,b]$, then by Lagrange's Mean Value Theorem, $\exists\, c \in (x_1,x_2)$

Such that $$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) > 0$$

or, $f(x_2)-f(x_1)>0$ or, $f(x_2)>f(x_1)$ for $x_2>x_1$

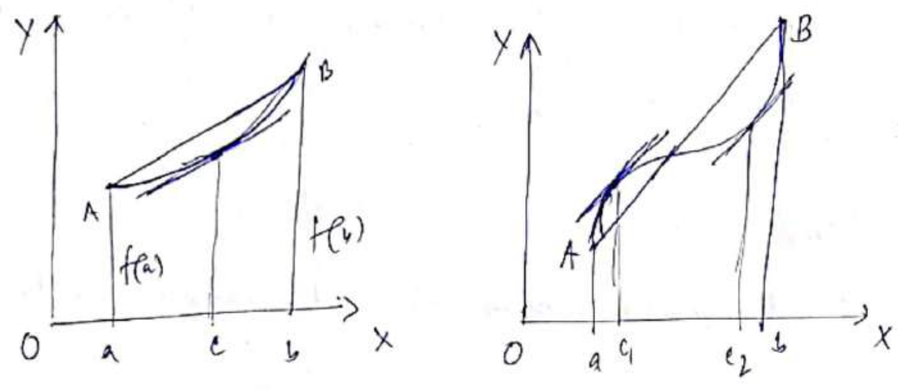
Thus, f is strictly increasing on $[a,b]$.

4. If a function f is (i) continuous on $[a,b]$, (ii) derivable on (a,b), and (iii) $f'(x)<0$, $\forall x \in (a,b)$, then f is strictly decreasing on $[a,b]$.

Proof: Exercise.

Geometrical Interpretation

The theorem simply states that between two points A and B of the graph of f there exists at least one point where the tangent is parallel to the chord AB.



5. If f' exists and is bounded on some interval I, then f is uniformly continuous

Proof: Let $x_1, x_2 \in I$ with $x_1 < x_2$.

Since f' is bounded on I, $\exists$ a positive real number K such that $|f'(x)| \le K$ $\forall x \in I$.

f satisfies both the condition of Mean Value Theorem on $[x_1, x_2]$ and therefore $\exists$ $c \in (x_1, x_2)$ such that

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c)$$

So, $$\left|\frac{f(x_2)-f(x_1)}{x_2-x_1}\right| = |f'(c)| \le K.$$ It follows that

$|f(x_2)-f(x_1)| \le K|x_2-x_1|$ $\forall x_1, x_2 \in I$. Let us choose $\varepsilon > 0$

Then $\exists$ a positive $\delta\left(=\frac{\varepsilon}{K}\right)$ such that

$|f(x_2)-f(x_1)|<\varepsilon \ \forall x_1, x_2 \in I$ satisfying $|x_2-x_1|<\delta$

Hence f is uniformly continuous on I.

Theorem 2.1.5 (Cauchy's Mean Value Theorem)

If ~~f~~ two real valued functions f, g defined on $[a,b]$ are

(i) continuous on $[a,b]$

(ii) derivable on (a,b), and

(iii) $g'(x) \neq 0$, for any $x \in (a,b)$,

then $\exists$ at least one real number ~~or between~~ $c \in (a,b)$ such that

that $$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$$

Proof: $g(a) \neq g(b)$. Because, if $g(a)=g(b)$ then g would satisfy all conditions of Rolle's theorem on $[a,b]$ and consequently $g'(x)$ would be equal to 0 for some point c in (a,b) and this would contradict the condition (iii) of the theorem.

Let us define a function $\phi:[a,b]\to\mathbb{R}$ by $\phi(x)=f(x)+A.g(x)$ $\forall x \in [a,b]$, ~~where~~ where A is a constant.

ϕ is continuous on $[a,b]$, since f and g are both continuous on $[a,b]$, ϕ is ~~differentiable~~ derivable on ~~[a,b]~~ (a,b), since f, g are both derivable on (a,b).

Let us choose A such that $\phi(a)=\phi(b)$.

Then $f(a)+A.g(a) = f(b)+A.g(b)$

or, $A=-\dfrac{f(b)-f(a)}{g(b)-g(a)}$ as $g(a)\neq g(b)$.

For this choice of A, ϕ satisfies all conditions of Rolle's theorem on $[a,b]$. So, $\exists$ a point $c \in (a,b)$ such that

$$\phi'(c)=0.$$