~~But $\phi'(c) = f'(c)$ $f(b)-f(a)$~~ But $\phi'(c) = f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c)$

and as $a < c < b$, $g'(c) \neq 0$, so,

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

This completes the proof.

Note 1 Lagrange's Mean Value Theorem can be deduced from Cauchy's Mean Value Theorem by taking $g(x) = x$, $x \in [a,b]$

Note 2 Both f and g satisfy the condition of Lagrange's Mean Value Theorem. Consequently, $\exists$ points c_1, d_1 in (a,b) such that $\frac{f(b)-f(a)}{b-a} = f'(c_1)$ $\frac{g(b)-g(a)}{b-a} = g'(d_1)$

c_1 and d_1 are different points in general and therefore a single point c in (a,b) may not be found to satisfy the conclusion of Cauchy's Mean Value Theorem, unless the third condition "$g'(x) \neq 0$ for all $x \in (a,b)$" is imposed on the function g.

Worked examples

1. Show that $\frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+u^2}$, if $0 < u < v$

And deduce that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

Solution: Let $f(x) = \tan^{-1}x$, then for x such that $u < x < v$, $f'(x) = \frac{1}{1+x^2}$. So f satisfies all conditions of mean value theorem. So, $\exists$ c such that $u < c < v$ and $\frac{\tan^{-1}v - \tan^{-1}u}{v-u} = \frac{1}{1+c^2}$

Now $c > u \Rightarrow \dfrac{1}{1+c^2} < \dfrac{1}{1+u^2}$ and $c < v \Rightarrow \dfrac{1}{1+c^2} > \dfrac{1}{1+v^2}$

So, $\dfrac{1}{1+v^2} < \dfrac{\tan^{-1}v - \tan^{-1}u}{v-u} < \dfrac{1}{1+u^2}$

or, $\dfrac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \dfrac{v-u}{1+u^2}$

The other result follows by taking $u=1$ and $v=\frac{4}{3}$

2. Show that $\dfrac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta$, for some θ such that $0 < \alpha < \theta < \beta < \pi/2$

Solution: Let $f(x) = \sin x$ and $g(x) = \cos x$ for $x \in [\alpha, \beta]$

f and g are both continuous and differentiable in $[\alpha, \beta]$ and $f'(x) = \cos x$ and $g'(x) = -\sin x \neq 0$. So, by Cauchy's mean value theorem on $[\alpha, \beta]$, $\exists\ \theta$ such

$$\frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta}, \quad \alpha < \theta < \beta$$

or, $$\frac{\sin\beta - \sin\alpha}{\cos\alpha - \cos\beta} = \cot\theta, \quad \alpha < \theta < \beta$$

3. A twice differentiable function f is such that $f(a) = f(b) = 0$ and $f(c) > 0$ for some real number c, such $a < c < b$. Prove that there at least one real value d between a and b for which $f''(d) < 0$

Solution: Let us consider the function f on $[a, b]$.. Since f'' exists, f' and f both exists and continuous on $[a, b]$. Since c is a point between a and b, applying Lagrange's Mean Value Theorem to f on the intervals $[a,c]$ and $[c,b]$ respectively, we get d_1 and d_2 such that

$$\frac{f(c)-f(a)}{c-a} = f'(d_1),\ a<d_1<b \text{ and } \frac{f(b)-f(c)}{b-c} = f'(d_2),\ c<d_2<b$$

but $f(a)=f(b)=0$, so $f'(d_1) = \frac{f(c)}{c-a}$, $f'(d_2) = -\frac{f(c)}{b-c}$ where

$a<d_1<c<d_2<b$.

Again $f'(x)$ is continuous and derivable on $[d_1, d_2]$. So, by Mean Value Theorem $\frac{f'(d_2)-f'(d_1)}{d_2-d_1} = f''(d)$ for some d such that $d_1<d<d_2$. Substituting the values of $f'(d_1)$ and $f'(d_2)$, we get,

$$f''(d) = -\frac{f(c)}{d_2-d_1}\left(\frac{1}{b-c}+\frac{1}{c-a}\right)$$
$$= -\frac{(b-a)f(c)}{(d_2-d_1)(b-c)(c-a)} < 0$$

4. If a function f is such that its derivative f' is continuous on $[a,b]$ and derivable on (a,b), then show that $\exists$ a number c between a and b such that $f(b) = f(a)+(b-a)f'(a)+\frac{1}{2}(b-a)^2 f''(c)$

Solution: Clearly the functions f and f' are continuous on $[a,b]$ and f is derivable on $[a,b]$ and f' is derivable on (a,b).

Consider the function $\phi(x) = f(b)-f(x)-(b-x)f'(x)-(b-x)^2A$ where A is a ~~const~~ constant to be determined such that $\phi(a)=\phi(b)$

So, $f(b)-f(a)-(b-a)f'(a)-(b-a)^2A = 0 \quad \ldots (1)$

Now, $\phi(x)$, being the sum of continuous and derivable functions, is itself continuous on $[a,b]$ and derivable on (a,b) and also $\phi(a)=\phi(b)$. So, ϕ satisfies all conditions of Rolle's theorem and therefore $\exists$ $c\in(a,b)$ such that

$\phi'(c)=0$

Now, $\phi'(x) = -f'(x)+f'(x)-(b-x)f''(x)+2(b-x)A$

So, $-(b-c)f''(c)+2(b-c)A = \phi'(c) = 0$

But $b-c\neq 0$, so $A=\frac{1}{2}f''(c)$ ... (2)

Hence, from (1) and (2),

$$f(b) = f(a)+(b-a)f'(a)+\frac{1}{2}(b-a)^2 f''(c)$$

5. Show that $\frac{\tan x}{x} > \frac{x}{\sin x}$, for $0<x<\pi/2$

Solution: So, we have to show that

$$\frac{\tan x}{x}-\frac{x}{\sin x}>0 \text{ or, } \frac{\sin x\tan x - x^2}{x\sin x}>0, \text{ for } 0<x<\pi/2$$

Since $x\sin x>0$ for $0<x<\pi/2$, it will therefore suffice to show that $\sin x\tan x - x^2>0$.

Let $f(x) = \sin x\tan x - x^2$. Then for $0<x<\pi/2$,

$$f'(x) = \cos x\tan x + \sin x\sec^2 x - 2x = \sin x + \sin x\sec^2 x - 2x$$

We can not decide about the sign of $f'(x)$ mainly because of the presence of $2x$ term.

The function $f'(x)$ is continuous and derivable on $[0,\pi/2]$

Now, $f''(x) = \cos x + \cos x\sec^2 x + 2\sin x\sec^2 x\tan x - 2$

$$= \left(\sqrt{\sec x}-\sqrt{\cos x}\right)^2 + 2\tan^2 x\sec x > 0, \text{ for } 0<x<\pi/2$$

So, $f'(x)$ is an increasing function. Further since $f'(0)=0$,