therefore $f'(x) > 0$ for $0 < x < \pi/2$

Again since $f'(x) > 0$, f is increasing function and because $f(0) = 0$, $f(x) > 0$ for $0 < x < \pi/2$

So, it follows that $\frac{\tan x}{x} > \frac{x}{\sin x}$ for $0 < x < \pi/2$

6. Prove that $\left(1+\frac{1}{x}\right)^x > \left(1+\frac{1}{y}\right)^y$ if $x, y \in \mathbb{R}$ and $x > y > 0$

Solution: Let $f(x) = \left(1+\frac{1}{x}\right)^x$, $x > 0$

Then $f'(x) = \left(1+\frac{1}{x}\right)^x \left[\log\left(1+\frac{1}{x}\right) + x \cdot \frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)\right]$

$$= \left(1+\frac{1}{x}\right)^x \left[\log\left(1+\frac{1}{x}\right) - \frac{\frac{1}{x}}{1+\frac{1}{x}}\right]$$

Let $\phi(x) = \log(1+x) - \frac{x}{1+x}$, $x \geq 0$

Then ϕ is continuous on $[0, \infty)$ and

$$\phi'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2} > 0 \text{ for } x > 0$$

Hence ϕ is strictly increasing function on $[0, \infty)$.

Since $\phi(0) = 0$, $\phi(x) > 0$ for $x > 0$. That is $\log\left(1+\frac{1}{x}\right) > \frac{x}{1+x}$ for $x > 0$. It follows that $\log\left(1+\frac{1}{x}\right) > \frac{\frac{1}{x}}{1+\frac{1}{x}}$ for $x > 0$

We also have $\left(1+\frac{1}{x}\right)^x > 0$ for $x > 0$. So, $f'(x) > 0$ for $x > 0$ showing that f is strictly increasing function for $x > 0$. Hence $x > y > 0 \Rightarrow \left(1+\frac{1}{x}\right)^x > \left(1+\frac{1}{y}\right)^y$.

7. A function $f: \mathbb{R} \to \mathbb{R}$ satisfies the condition $|f(x) - f(y)| \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function on $\mathbb{R}$.

Solution: By the condition,

$$-(x-y)^2 \leq f(x) - f(y) \leq (x-y)^2 \text{ for all } x, y \in \mathbb{R}.$$

Let $c \in \mathbb{R}$. Then $-h^2 \le f(c+h) - f(c) \le h^2$ for all $h \in \mathbb{R}$.

That is, $-h \le \frac{f(c+h)-f(c)}{h} \le h$ if $h>0$ and

$$h \le \frac{f(c+h)-f(c)}{h} \le -h \quad \text{if } h<0.$$

By Sandwich theorem, it follows from the first that

$\lim_{h\to 0+} \frac{f(c+h)-f(c)}{h} = 0$ and it follows from the second

that $\lim_{h\to 0-} \frac{f(c+h)-f(c)}{h} = 0$

Consequently $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = 0$. That is, $f'(c) = 0$ for all $c \in \mathbb{R}$

This proves that f is a constant function on $\mathbb{R}$.

Theorem 2.1.6 ~~If~~ (Taylor's Theorem)

If a function f defined on $[a, a+h]$, is such that

(i) the $(n-1)$th derivative f^{n-1} is continuous on $[a, a+h]$ and

(ii) the nth derivative f^n exists in $(a, a+h)$, then $\exists$ at least one real number θ, ~~such that~~ $0<\theta<1$, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n(1-\theta)^{n-p}}{p\,[(n-1)!]}f^n(a+\theta h) \quad \cdots (1)$$

where p is a given positive integer.

Proof: First of all we observe that the condition (i) in the statement implies that all the derivatives $f', f'', \ldots, f^{n-1}$ exists and are continuous on $[a, a+h]$. Consider the function ϕ defined ~~by~~ on $[a, a+h]$ as

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \cdots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x) + A(a+h-x)^p$$

where A is a constant to be determined such that $\phi(a) = \phi(a+h)$

So, $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + Ah^p \quad \cdots (2)$

Now, (i) $f, f', f'', \ldots, f^{n-1}$ being all continuous on $[a, a+h]$, the function $\phi(x)$ is continuous on $[a, a+h]$

(ii) the functions, $f, f', f'', \ldots f^{n-1}$ and $(a+h-x)^r$ for all r being derivable in $(a, a+h)$, the function ϕ is derivable on $(a, a+h)$

and (iii) $\phi(a) = \phi(a+h)$

Thus, the function ϕ satisfies all the conditions of Rolle's theorem and hence $\exists$ at least one real number θ, $0<\theta<1$ such that $\phi'(a+\theta h) = 0$.

But $\phi'(x) = \dfrac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - Ap(a+h-x)^{p-1}$

$$\therefore \quad 0 = \phi'(a+\theta h) = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h) - Ap\,h^{p-1}(1-\theta)^{p-1}$$

$$\Rightarrow \quad A = \frac{h^{n-p}(1-\theta)^{n-p} f^n(a+\theta h)}{p\,[(n-1)!]}, \quad h\neq 0,\ \theta\neq 1 \qquad \text{---}(2)$$

Substituting A in (1) from (2), we get (1) as

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n(1-\theta)^{n-p} f^n(a+\theta h)}{p\,[(n-1)!]}$$

Forms of Remainder after n terms

(i) The term $R_n = \dfrac{h^n(1-\theta)^{n-p}}{p\,[(n-1)!]} f^n(a+\theta h)$

which occurs after n terms, is known as Taylor's remainder after n terms. The theorem with this form of remainder is known as Taylor's Theorem with Schlömilch and Röche form of remainder.

(ii) For $p=1$, we get $R_n = \dfrac{h^n(1-\theta)^{n-1} f^n(a+\theta h)}{(n-1)!}$ which is called Cauchy's form of remainder.

(iii) For $p=n$, we get $R_n = \dfrac{h^n}{n!} f^n(a+\theta h)$ which is called Lagrange's form of remainder.

Second form of Taylor's Theorem : If f satisfies the conditions of Taylor's theorem in $[a, a+h]$ and x is any point of $[a, a+h]$ then it satisfies the conditions in the interval $[a, x]$ also. Replacing $a+h$ by x or h by $(x-a)$ in (1), we get

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(x-a)^n (1-\theta)^{n-p}}{p[(n-1)!]} f^n(a + \theta(x-a)) \quad \dots (4)$$

where $0<\theta<1$.

The remainder after ~~n~~ n terms can thus be written as

$$R_n = \frac{(x-a)^n (1-\theta)^{n-p}}{p[(n-1)!]} f^n(c)$$ where c lies between a and x

and depends on the selection of x.

Theorem 2.1.7 (Maclaurin's Theorem)

Putting $a=0$, in (4), we have for $x \in [0, h]$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n (1-\theta)^{n-p}}{p[(n-1)!]} f^n(\theta x)$$

is called Maclaurin's Theorem with Schlömilch & Röche form remainder.

Cauchy's form of remainder (for $p=1$)

$$R_n = \frac{x^n (1-\theta)^{n-1}}{(n-1)!} f^n(\theta x)$$

Lagrange's form of remainder (for $p=n$)

$$R_n = \frac{x^n}{n!} f^n(\theta x)$$

We have thus proved Maclaurin's Theorem. Thus Maclaurin's Theorem with Lagrange's form of remainder may be stated as : If f^{n-1} is continuous in $[0, h]$ and is derivable in $(0, h)$, then for each $x \in [0, h]$, there exists a number θ, $0<\theta<1$ such that $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x)$.

Theorem 2.1.8 (Generalised Mean Value Theorem (Taylor's Theorem))
[Deduction of Taylor's Theorem from the Mean Value Theorem]