Let a function f be such that $(n-1)$th derivative f^{n-1} is continuous in $[a, a+h]$ and its nth derivative f^n exists in $(a, a+h)$.

Consequently, the functions $f, f', f'', \dots, f^{n-1}$ exists and are continuous in $[a, a+h]$ while f^n exists in ~~(a, a+h)~~ $(a, a+h)$

Consider the function $$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \dots$$
$$\dots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x)$$

which, being the sum of continuous and derivable functions, is itself ~~is~~ is continuous in $[a, a+h]$ and derivable in $(a, a+h)$. So, by Lagrange's mean value Theorem $\exists$ a positive number θ between 0 and 1 such that $$\phi(a+h) = \phi(a) + h\,\phi'(a+\theta h)$$

now $$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x)$$

So, $$\phi'(a+\theta h) = \cancel{\frac{(a+h-x)^{n-1}}{(n-1)!} f^n(a+\theta h)} \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$$

Also $$\phi(a) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a)$$

and $\phi(a+h) = f(a+h)$

So, $$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$$

Where $0 < \theta < 1$, which is Taylor's Theorem with Cauchy's form of remainder.

Taylor's Infinite series and power series expansions.

We have seen that
$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + R_n \quad \dots (5)$$
where R_n is the remainder after n terms.

The result can be interpreted in two ways :

(i) The value $f(a+h)$ of the function at a point may be approximated by a summation of the terms like $\frac{h^r}{r!} f^r(a)$ involving values of the function and its derivatives at some other point of the domain of definition.

(ii) The value $f(a+h)$ ~~of the~~ of the function may be expanded in powers of h

The natural question as to how far the R.H.S of (5) correctly represents the L.H.S is answered if the series $f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots$ converges to $f(a+h)$

Let $S_n = f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)$ so that

$$f(a+h) = S_n + R_n$$

Thus if $S_n \to 0$ as $n\to\infty$, we have

$$\lim_{n\to\infty} S_n = f(a+h)$$

i.e the infinite series

$$f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\cdots$$

Converges to $f(a+h)$

Thus we have proved that if a function f possesses derivatives of every order in $[a, a+h]$ and Taylor's remainder $R_n \to 0$ as $n\to\infty$, then

$$f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots = f(a+h)$$

The infinite series
$$f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\cdots \quad (6)$$
is called Taylor's series. It can be looked upon as expansion of $f(a+h)$ in powers of h

Similarly for $x\in[a, a+h]$, when $\lim_{n\to\infty} R_n = 0$, we have from equation (4) (page-28)

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots \quad (7)$$

which is the expansion of $f(x)$ in powers of $(x-a)$

Maclaurin's Infinite series

we may easily deduce from (5') or (6) that if f possesses derivative of every order in $[0, h]$ and $\lim_{n\to\infty} R_n = 0$, then for all $x \in [0,h]$,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

which is Maclaurin's infinite series expansion of $f(x)$ in powers of x.

Note: In the above discussion, R_n can be in any of the forms.

Example 8. Show that the number θ which occurs in the Taylor's theorem with Lagrange's form of remainder after n terms approaches to the limit $\frac{1}{n+1}$ as h approaches zero, provided that $f^{n+1}(x)$ is continuous and different from zero at $x=a$

Proof: Applying Taylor's theorem with remainder after n terms and $n+1$ terms successively, we get

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h)$$

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^n}{n!} f^n(a) + \frac{h^{n+1}}{(n+1)!} f^{n+1}(a+\theta' h)$$

$$0<\theta<1, \quad 0<\theta'<1$$

These give

$$\frac{h^n}{n!} f^n(a+\theta h) = \frac{h^n}{n!} f^n(a) + \frac{h^{n+1}}{(n+1)!} f^{n+1}(a+\theta' h)$$

or, $f^{n}(a+\theta h) - f^{n}(a) = \frac{h}{n+1} f^{n+1}(a+\theta' h)$

Applying Lagrange's Mean Value theorem to the left side, we have

$$\theta h f^{n+1}(a+\theta''\theta h) = \frac{h}{n+1} f^{n+1}(a+\theta' h) \quad 0<\theta''<1$$

or, $\theta = \frac{1}{n+1} \frac{f^{n+1}(a+\theta' h)}{f^{n+1}(a+\theta''\theta h)}$

Taking limit when $h \to 0$, we get

$$\lim_{h\to 0} \theta = \frac{1}{n+1}$$

Expansion of e^x, $\log(1+x)$, $(1+x)^m$, $\sin x$ and $\cos x$ with their range of validity (using Maclaurin's series).

1. Let $f(x) = e^x$ so that $f^{n}(x) = e^x \ \forall n$.

Evidently $f(x)$ and all its derivatives exist and are continuous for every real value of x. Let us now consider the limit of the remainder R_n

Taking Lagrange's form of the remainder, we have

$$R_n = \frac{x^n}{n!} f^{n}(\theta x) = \frac{x^n}{n!} e^{\theta x}, \quad 0<\theta<1$$

So, $\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{x^n}{n!} e^{\theta x} = e^{\theta x}\left(\lim_{n\to\infty} \frac{x^n}{n!}\right) = 0$

Thus the condition of Maclaurin's infinite expansion are satisfied. Now $f(0) = 1$, $f^{n}(0) = 1$ for all integral values of n.

Substituting these values in the Maclaurin's infinite series, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \forall x \in \mathbb{R}.$$