2. Let $f(x) = \sin x$, $x \in \mathbb{R}$. $f^n(x) = \sin\left(\frac{n\pi}{2} + x\right)$ $\forall n \in \mathbb{N}$

Evidently $f(x)$ and all its derivatives exist and are continuous for every real value of x. By Taylor's theorem with Lagrange's form of remainder after n terms, for any non-zero $x \in \mathbb{R}$,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + R_n(x), \text{ where}$$

$$R_n(x) = \frac{x^n f^n(\theta x)}{n!} \text{ for some real number } \theta \text{ satisfying } 0<\theta<1 \quad \text{--(i)}$$

$$f(0) = 0, \quad f^n(0) = \sin\frac{n\pi}{2} = 0 \text{ if } n \text{ be even}$$

$= 1$ if $n = 4k+1$, k being an integer

$= -1$ if $n = 4k+3$, k being an integer

So, $f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{x^n}{n!}\sin\left(\frac{n\pi}{2} + \theta x\right)$

now $|R_n| = \left|\frac{x^n}{n!}\sin\left(\frac{n\pi}{2} + \theta x\right)\right| \leq \left|\frac{x^n}{n!}\right|$

So, $\lim_{n\to\infty} R_n = 0$

Thus the condition of MacLaurin's infinite expansion is satisfied

So, we get

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots$ for all real x

as convergence at $x=0$ holds trivially.

3. Let $f(x) = \cos x$, $x \in \mathbb{R}$

Do it yourself as exercise (it is similar to 2.)

4. Let $f(x) = \log(1+x)$, $-1 < x \le 1$

For all $n \in \mathbb{N}$, $f^n(x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

By Taylor's theorem for any real non-zero $x > -1$,

$f(x) = f(0) + x f'(0) + \dfrac{x^2}{2!} f''(0) + \cdots + \dfrac{x^{n-1}}{(n-1)!} f^{n-1}(0) + R_n$, where

$R_n = \dfrac{x^n}{n!} f^n(\theta x)$ (Lagrange's form) for some real θ satisfying $0<\theta<1$,

$R_n = \dfrac{x^n (1-\theta)^{n-1}}{(n-1)!} f^n(\theta x)$ (Cauchy's form) for some real θ satisfying $0<\theta<1$.

But $f(0) = 0$, $f^n(0) = (-1)^{n-1}(n-1)!$, $f^n(\theta x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n}$

So, $f(x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots + (-1)^{n-2}\dfrac{x^{n-1}}{n-1} + R_n$ where

$R_n = \dfrac{(-1)^{n-1}}{n} \dfrac{x^n}{(1+\theta x)^n}$ (Lagrange's form),

$R_n = (-1)^{n-1}(1-\theta)^{n-1}\dfrac{x^n}{(1+\theta x)^n}$ (Cauchy's form)

Case I Let $0 < x \le 1$. We take R_n in Lagrange's form

$R_n = \dfrac{(-1)^{n-1}}{n} \dfrac{x^n}{(1+\theta x)^n}$

In $0<x<1$, $1+\theta x > x > 0$. So, $0 < \dfrac{x}{1+\theta x} < 1$

When $x=1$, $\dfrac{x}{1+\theta x} = \dfrac{1}{1+\theta} < 1$

So, in $0 < x \le 1$, $0 < \dfrac{x}{1+\theta x} < 1$ and hence $0 < \left(\dfrac{x}{1+\theta x}\right)^n < 1$

So, $\lim\limits_{n\to\infty} |R_n| = \lim\limits_{n\to\infty} \dfrac{1}{n}\left|\dfrac{x}{1+\theta x}\right|^n = 0$, Since $\lim\limits_{n\to\infty}\dfrac{1}{n} = 0$ and

$\left|\dfrac{x}{1+\theta x}\right|^n$ is bounded. Hence $\lim\limits_{n\to\infty} R_n = 0$

Case 2. Let $-1 < x < 0$. We take R_n in Cauchy's form.

Here $R_n = (-1)^{n-1}(1-\theta)^{n-1}\dfrac{x^n}{(1+\theta x)^n}$

So, $|R_n| = |x|^n \left|\dfrac{1-\theta}{1+\theta x}\right|^{n-1} \cdot \dfrac{1}{|1+\theta x|}$

In $-1 < x < 0$, $0 < 1-\theta < 1+\theta x < 1$

So, $0 < \dfrac{1-\theta}{1+\theta x} < 1$ and hence and hence $0 < \left(\dfrac{1-\theta}{1+\theta x}\right)^{n-1} < 1$

In $-1 < x < 0$, $\lim\limits_{n\to\infty} |x|^n = 0$

For all real x, $-|x| \le x \le |x|$. ~~Hence~~ $\cancel{-\theta|x| \le \theta x}$

Hence $-|x| < -\theta|x| \le \theta x \le \theta|x| < |x|$ since $0<\theta<1$.

In $-1 < x < 0$, $0 < 1-|x| < 1+\theta x$. So, $\dfrac{1}{|1+\theta x|} < \dfrac{1}{1-|x|}$

Hence $\lim\limits_{n\to\infty} |R_n| = 0$ and this implies $\lim\limits_{n\to\infty} R_n = 0$

So, the infinite ~~set~~ series $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dots$ converges to $\log(1+x)$ for all non-zero $x \in (-1, 1]$.

At $x=0$, the convergence holds trivially.

So, $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dots$ for all $x \in (-1, 1]$

5. Let $f(x) = (1+x)^m$, $x \in \mathbb{R}$

Case 1 Let m be a positive integer

Then $f^n(x) = m(m-1)\dots(m-n+1)(1+x)^{m-n}$ if $1 \le n < m$
$= m!$ if $n = m$
$= 0$ if $n > m$

By Taylor's theorem with remainder after m+1 terms, for any non-zero $x \in \mathbb{R}$,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^m}{m!} f^m(0)$$

But $f(0) = 1$, $f^n(0) = {}^mc_n$ for $1 \leq n \leq m$. So,

$$(1+x)^m = 1 + {}^mc_1 x + {}^mc_2 x^2 + \cdots + {}^mc_m x^m \text{ for all non-zero } x \in \mathbb{R}$$

At $x=0$, the equality holds trivially. So,

$$(1+x)^m = 1 + {}^mc_1 x + {}^mc_2 x^2 + \cdots + {}^mc_m x^m \text{ for all } x \in \mathbb{R}.$$

Thus we obtain a finite series expansion in this case.

Case 2 Let m be not a positive integer

In this case f is defined for all $x \neq -1$, if m be a negative integer and f is defined for all $x > -1$, if m be not an integer,

Considering all cases, f is defined for all $x > -1$, and for all $n \in \mathbb{N}$, $f^n(x) = m(m-1)\cdots(m-n+1)(1+x)^{m-n}$ for all $x > -1$

By Taylor's theorem with Cauchy's form of remainder after n terms, for any real non-zero $x > -1$,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + R_n \text{ where}$$

$$R_n = \frac{x^n (1-\theta)^{n-1}}{(n-1)!} f^n(\theta x) \text{ for some real } \theta \text{ satisfying } 0 < \theta < 1 \quad \cdots \text{ (i)}$$

But $f(0) = 1$, $f^n(0) = m(m-1)\cdots(m-n+1)$,

$$f^n(\theta x) = m(m-1)\cdots(m-n+1)(1+\theta x)^{m-n}$$

$$\text{So, } f(x) = 1 + mx + \frac{m(m-1)}{2!} x^2 + \cdots + \frac{m(m-1)\cdots(m-n+1)}{(n-1)!} x^{n-1}$$

$$+ R_n \text{ where } R_n = \frac{x^n (1-\theta)^{n-1}}{(n-1)!} m(m-1)\cdots(m-n+1)(1+\theta x)^{m-n}$$