Now let r_j denote the number of dots in the jth row of the dot diagram. Observe that $r_1 \geq r_2 \geq \cdots \geq r_{p_1}$. Furthermore, the diagram can be reconstructed from the values of r_i's. ~~The proofs of these facts~~

In example 1, with $n_i = 4$, $p_1 = p_2 = 3$, $p_3 = 2$ and $p_4 = 1$, the dot diagram of T_i is as follows:

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & & \end{array}$$

Here $r_1 = 4$, $r_2 = 3$ and $r_3 = 2$

We now devise a method for computing the dot ~~diagra~~ diagram of T_i using the ranks of ~~the~~ linear operators determined by T and λ_i. Hence the dot diagram is completely determined by T, from which it follows that it is unique. On the other hand, β_i is not unique. We can give example for that (It is for this reason that we associate the dot diagram with T_i rather than with β_i).

To determine the dot diagram of T_i, we devise a method for computing each r_j, the number of dots in the jth row of the dot diagram, using only T and λ_i. The next three results give us the required method. To facilitate our arguments, we fix a basis β_i for K_{λ_i} so that β_i is a disjoint union of n_i cycles of generalized eigenvectors with lengths $p_1 \geq p_2 \geq \cdots \geq p_{n_i}$.

Theorem 7.2.1 For any positive integer r, the vectors in β_i that are associated with the dots in the first r rows of the dot diagram of T_i constitute a basis for $N((T-\lambda_i I)^r)$.
Hence the number of dots in the first r rows of the dot diagram equals nullity$((T-\lambda_i I)^r)$.

Proof: Clearly, $N((T-\lambda_i I)^r) \subseteq K_{\lambda_i}$ and K_{λ_i} is invariant under $(T-\lambda_i I)^r$. Let U denote the restriction of $(T-\lambda_i I)^r$ to K_{λ_i}. By the preceeding remarks,
$N((T-\lambda_i I)^r) = N(U)$, and hence it suffices to establish the theorem for U. Now define

$$S_1 = \{x \in \beta_i : U(x) = \theta\} \text{ and } S_2 = \{x \in \beta_i : U(x) \neq \theta\}$$

Let a and b denote the number of vectors in S_1 and S_2, respectively and let $m_i = \dim(K_{\lambda_i})$. Then $a+b = m_i$. For any $x \in \beta_i$, $x \in S_1$ if and only if x is one of the first r vectors of a cycle, and this is true if and only if x corresponds to a dot in the first r rows of the dot diagram. Hence a is the number of dots in the first r rows of the dot diagram. For any $x \in S_2$, the effect of applying U to x is to move the dot corresponding to x exactly r places up its column to another dot. It follows that U maps S_2 in a one-to-one fashion into β_i. Thus $\{U(x) : x \in S_2\}$ is a basis for $R(U)$ consisting of b vectors. Hence rank$(U) = b$, and so nullity$(U) = m_i - b = a$. But S_1 is a linearly independent subset of $N(U)$ consisting of a vectors; therefore S_1 is a basis for $N(U)$.

In the case that $r=1$, Theorem 7.2.1 yeilds the following Corollary:

Corollary 7.2.2: The dimension of E_{λ_i} is n_i. Hence in a Jordan Canonical form of T, the number of Jordan blocks corresponding to λ_i equals the dimension of E_{λ_i}.

Proof: Exercise.

We are now able to devise a method for describing the dot diagram in terms of the ranks of the operators.

Theorem 7.2.3 Let r_j denote the number of dots in the jth row of the dot diagram of T_i, the restriction of T to K_{λ_i}. Then the following statements are true:

(a) $r_1 = \dim(V) - \operatorname{rank}(T - \lambda_i I)$

(b) $r_j = \operatorname{rank}\left((T-\lambda_i I)^{j-1}\right) - \operatorname{rank}\left((T-\lambda_i I)^{j}\right)$ if $j>1$

Proof: By Theorem 7.2.1, for $1 \le j \le p_1$, we have

$$\begin{aligned} r_1 + r_2 + \cdots + r_j &= \operatorname{nullity}\left((T-\lambda_i I)^j\right) \\ &= \dim(V) - \operatorname{rank}\left((T-\lambda_i I)^j\right). \end{aligned}$$

Hence $r_1 = \dim(V) - \operatorname{rank}\left((T-\lambda_i I)\right)$

and for $j>1$

$$\begin{aligned} r_j &= (r_1 + r_2 + \cdots + r_j) - (r_1 + r_2 + \cdots + r_{j-1}) \\ &= \left\{\dim(V) - \operatorname{rank}\left((T-\lambda_i I)^j\right)\right\} \\ &\quad - \left\{\dim(V) - \operatorname{rank}\left((T-\lambda_i I)^{j-1}\right)\right\} \end{aligned}$$

$= \operatorname{rank}\left((T-\lambda_i I)^{j-1}\right) - \operatorname{rank}\left((T-\lambda_i I)^{j}\right).$

Theorem 7.2.3 shows that the dot diagram of T_i is completely determined by T and λ_i. Hence we have proved the following result:

Corollary 7.2.4 For any eigenvalue λ_i of T, the dot diagram of T_i is unique. Thus, subject to the convention that the cycles of generalized eigenvectors for the bases of each generalized eigenspace are listed in order of decreasing length, the Jordan canonical form of a linear operator or a matrix is unique upto the ordering ~~...~~ of the eigenvalues.

We apply these results to find the Jordan Canonical forms of two matrices and a linear operator.

Example 2 Let $A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}$

We find the Jordan Canonical form of A and a Jordan canonical basis for the linear operator $T = L_A$.

The characteristic polynomial of A is

$$\det(A - tI) = (t-2)^3(t-3).$$

Thus A has two distinct eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ with multiplicities 3 and 1, respectively. Let T_1 and T_2 be the restriction of L_A to the generalized eigenspaces K_{λ_1} and K_{λ_2} respectively.

Scanned with CamScanner