Suppose that β_1 is a Jordan canonical basis for T_1. Since λ_1 has multiplicities, it follows that $\dim(K_{\lambda_1}) = 3$ by Theorem 7.1.6(c) (page-142; has the dot diagram of T_1 has three dots. As we did earlier, let r_j denote the number of dots in the jth row of this dot diagram. Then, by Theorem 7.2.3,

$$r_1 = 4 - \operatorname{rank}(A-2I) = 4 - \operatorname{rank}\begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = 4-2=2$$

and $r_2 = \operatorname{ran}(A-2I) - \operatorname{rank}\left((A-2I)^2\right) = 2-1=1$

(Actually, the computation of r_2 is unnecessary in this case because $r_1 = 2$ and the dot diagram only three dots)

Hence the dot diagram associated with β_1 is

$$\begin{array}{cc} \bullet & \bullet \\ \bullet & \end{array}$$

So, $$A_1 = [T_1]_{\beta_1} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Since $\lambda_2 = 3$ has multiplicity 1, it follows that $\dim(K_{\lambda_2}) = 1$ and consequently any basis β_2 of K_{λ_2} consists of a single eigenvector corresponding to $\lambda_2 = 3$. Therefore $A_2 = [T_2]_{\beta_2} = (3)$

Setting $\beta = \beta_1 \cup \beta_2$, we have

$$J = [L_A]_\beta = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

and so J is the Jordan canonical form of A.

We now find a Jordan Canonical basis for $T = L_A$. We begin by determining a Jordan canonical basis β_1 for T_1. Since the dot diagram of T_1 has two columns, each corresponding to a cycle of generalized eigenvectors, there are two such cycles. Let v_1 and v_2 denote the end vectors of the first and second cycles, respectively. We reprint below the dot diagram with the dots labeled with the names of the vectors to which they correspond.

$$\begin{array}{ll} \bullet\ (T-2I)(v_1) & \bullet\ v_2 \\ \bullet\ v_1 & \end{array}$$

From this diagram we see that $v_1 \in N\left((T-2I)^2\right)$ but $v_1 \notin N\left((T-2I)\right)$. Now

$$A-2I = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \text{ and } (A-2I)^2 = \begin{pmatrix} 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 \end{pmatrix}$$

It is easily seen that

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

is a basis for $N\left((T-2I)^2\right) = K_{\lambda_1}$. Of these three vectors, the last two do not belong to $N(T-2I)$, and hence we select one of these for v_1. Suppose

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \quad \text{Then } (T-2I)(v_1) = (A-2I)(v_1)$$

$$= \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

Now simply choose v_2 to be a vector in E_{λ_1} that is linearly independent of $(T-2I)(v_1)$; for example, select

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus we have associated the Jordan canonical basis

$$\beta_1 = \left\{ \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

with the dot diagram in the following manner

$$\begin{array}{ll} \bullet \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} & \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \bullet \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} & \end{array}$$

By Theorem 7.1.10 (page-146), the linear independence of β_1 is guaranteed since v_2 was chosen to be linearly independent of $(T-2I)(v_1)$.

Since $\lambda_2 = 3$ has multiplicity 1, $\dim(K_{\lambda_2}) = \dim(E_{\lambda_2}) = 1$. Hence any eigenvector of L_A corresponding to $\lambda_2 = 3$ constitutes an appropriate basis ~~for~~ β_2. For example,

$$\beta_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Thus $$\beta = \beta_1 \cup \beta_2 = \left\{ \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a Jordan canonical basis for L_A.

notice that if
$$Q = \begin{pmatrix} -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \quad \text{then } J = Q^{-1} A Q.$$

Example 3 Let $A = \begin{pmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{pmatrix}$.

We find the Jordan canonical form J of A, a Jordan canonical basis for L_A, and a matrix Q such that
$$J = Q^{-1} A Q$$

The characteristic polynomial of A is $\det(A - tI)$ $= (t-2)^2 (t-4)^2$. Let $T = L_A$, $\lambda_1 = 2$ and $\lambda_2 = 4$, and let ~~T~~ T_i be the restriction of L_A to K_{λ_i} for $i = 1, 2$

We begin by computing the dot diagram of T_1. Let r_1 denote the number of dots in the first row of this diagram. Then
$$r_1 = 4 - \operatorname{rank}(A - 2I) = 4 - 2 = 2;$$
hence the dot ~~dog~~ diagram for T_1 is as follows

$$\bullet \quad \bullet$$

Therefore $A_1 = [T_1]_{\beta_1} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$,

where β_1 is any basis corresponding to the dots. In this case, β_1 is an arbitrary basis for $E_{\lambda_1} = N(T - 2I)$, for example,
$$\beta_1 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \right\}.$$