Next we compute the dot diagram of T_2. Since $\text{rank}(A-4I)=3$, there is only $4-3=1$ dot in the first row of the diagram. Since $\lambda_2=4$ has multiplicity 2, we have $\dim(K_{\lambda_2})=2$, and hence this dot diagram has the following form:

$$\begin{matrix}\bullet\\ \bullet\end{matrix}$$

Then $A_2=[T_2]_{\beta_2}=\begin{pmatrix}4 & 1\\ 0 & 4\end{pmatrix}$ where β_2 is any basis for K_{λ_2} corresponding to the dots. In this case, β_2 is a cycle of length 2. The end vector of this cycle is a vector $v\in K_{\lambda_2}=N((T-4I)^2)$ such that $v\notin N(T-4I)$. One way finding such a vector was used to select v_1 in Example 2. In this example, we illustrate another method. A simple calculation shows that a basis for the null space of L_A-4I is

$$\left\{\begin{pmatrix}0\\1\\1\\1\end{pmatrix}\right\}$$

Choose v to be any solution to the system of linear equations $(A-4I)x=\begin{pmatrix}0\\1\\1\\1\end{pmatrix}$,

for example, $v=\begin{pmatrix}1\\-1\\-1\\0\end{pmatrix}$

Thus $\beta_2=\{(L_A-4I)(v), v\}=\left\{\begin{pmatrix}0\\1\\1\\1\end{pmatrix},\begin{pmatrix}1\\-1\\-1\\0\end{pmatrix}\right\}$

Therefore $\beta = \beta_1 \cup \beta_2 = \left\{ \begin{pmatrix} 2\\1\\0\\2 \end{pmatrix}, \begin{pmatrix} 0\\1\\2\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\1\\1 \end{pmatrix}, \begin{pmatrix} 1\\-1\\-1\\0 \end{pmatrix} \right\}$

is a Jordan Canonical basis for L_A. The corresponding Jordan Canonical form is given by

$$J = [L_A]_\beta = \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix} = \left(\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right)$$

Finally, we define $Q = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 0 & 2 & 1 & -1 \\ 2 & 0 & 1 & 0 \end{pmatrix}$

Then $J = Q^{-1} A Q$.

<u>Example 4</u> Let V be a vector space of polynomial functions in two real variables x and y of degree at most 2. Then V is a vector space over $\mathbb{R}$ and $\alpha = \{1, x, y, x^2, y^2, xy\}$ is ~~an~~ an ordered basis for V. Let T be a linear operator on V defined by $T(f(x,y)) = \frac{\partial}{\partial x}(f(x,y))$

For example, if $f(x,y) = x + 2x^2 - 3xy + y$

$$T(f(x,y)) = \frac{\partial}{\partial x}\left(x + 2x^2 - 3xy + y\right) = 1 + 4x - 3y$$

We find the Jordan canonical form and a Jordan canonical basis for T. Let $A = [T]_\alpha$. Then

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and hence the characteristic polynomial of T is

$$\det(A-tI) = \det\begin{pmatrix} -t & 1 & 0 & 0 & 0 & 0 \\ 0 & -t & 0 & 2 & 0 & 0 \\ 0 & 0 & -t & 0 & 0 & 1 \\ 0 & 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 0 & -t & 0 \\ 0 & 0 & 0 & 0 & 0 & -t \end{pmatrix} = t^6$$

Thus $\lambda = 0$ is the only eigenvalue of T, and $K_\lambda = V$.

For each j, let r_j denote the number of dots in the jth row of the dot diagram of. By Theorem 7.2.3

$r_1 = 6 - \text{rank} A = 6 - 3 = 3$

and since $A^2 = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$r_2 = \text{rank}(A) - \text{rank}(A^2) = 3 - 1 = 2$

Because there are a total of six dots in the dot diagram and $r_1 = 3$ and $r_2 = 2$, it follows that $r_3 = 1$. So, the dot diagram of T is

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \end{array}$$

We conclude that the Jordan Canonical form of T is

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We now find a Jordan canonical basis for T. Since, the first column of the dot diagram of T consists of three dots, we must find a polynomial $f_1(x,y)$ such that $\frac{\partial^2}{\partial x^2}(f_1(x,y)) \neq \theta$. Examining the basis $\alpha = \{1, x, y, x^2, y^2, xy\}$ for $K_\lambda = V$, we see that x^2 is a suitable candidate.

Setting $f_1(x,y) = x^2$, we see that

$$(T-\lambda I)(f_1(x,y)) = T(f_1(x,y)) = \frac{\partial}{\partial x}(x^2) = 2x$$

and $$(T-\lambda I)^2(f_1(x,y)) = T^2(f_1(x,y)) = \frac{\partial^2}{\partial x^2}(x^2) = 2$$

Likewise, since the second column of the dot diagram consists of two dots, we must find a polynomial $f_2(x,y)$ such that

$$\frac{\partial}{\partial x}(f_2(x,y)) \neq \theta, \text{ but } \frac{\partial^2}{\partial x^2}(f_2(x,y)) = \theta$$

Since our choice must be linearly independent of the polynomials already chosen for the first cycle, the only choice in α that satisfies ~~this~~ these constraints is xy. So, we ~~set~~ set $f_2(x,y) = xy$. Thus

$$(T-\lambda I)(f_2(x,y)) = T(f_2(x,y)) = \frac{\partial}{\partial x}(xy) = y$$

Finally, the third column of the dot diagram consists of a single polynomial that lies in the null space of T. The only ~~notable~~ remaining polynomial in α is y^2, and it is suitable here. So, set $f_3(x,y) = y^2$. Therefore we have identified polynomials with the dots in the dot diagram as follows: