$\bullet\, z \quad \bullet\, y \cdot \bullet\, y^2$

$\bullet\, 2x \quad \bullet\, zy$

$\bullet\, x^2$

Thus $\beta = \{z, 2x, x^2, y, xy, y^2\}$ is a Jordan canonical basis for T.

7.3. Rational Canonical form

Before going to the discussion of Rational Canonical form we give some definition as follows:

Definition 7.3.1: Let T be a linear operator on a finite dimensional vector space V, and let x be a non-zero vector in V. The polynomial $p(t)$ is called a T-annihilator ~~T-annihilator~~ of x if $p(t)$ is a monic polynomial of least degree for which $p(T)(x) = \theta$.

Theorem 7.3.2 Let T be a linear operator on a finite dimensional vector space V, and let x be a non-zero vector in V. Then

(a) The vector x has a unique T-annihilator

(b) ~~If p(t) is the T-annihilator~~ The T-annihilator of x divides any polynomial $g(t)$ for which $g(T) = T_\theta$

(c) If $p(t)$ is the T-annihilator of x and W is the T-cyclic subspace generated by x, then $p(t)$ is the minimal polynomial of T_W, and $\dim(W)$ equals the degree of $p(t)$.

(d) The degree of the T-annihilator of x is 1 if and only if x is an eigenvector of T

Proof: Exercise

Until now, we have used eigenvalues, eigenvectors and generalized eigenvectors in our analysis of linear operators with characteristic polynomials that split. In general, characteristic polynomials need not split and operators need not have eigenvalues. However, unique factorization theorem guarantees that the characteristic polynomial $f(t)$ of any linear operator T on an n dimensional vector space factors uniquely as

$$f(t) = (-1)^n (\phi_1(t))^{n_1} (\phi_2(t))^{n_2} \cdots (\phi_k(t))^{n_k},$$

where the $\phi_i(t)$'s, $i = 1, 2, \ldots, k$ are distinct irreducible monic polynomial and the n_i's are positive integers. In the case that $f(t)$ splits, each irreducible monic polynomial factor is of the form $\phi_i(t) = t - \lambda_i$, where λ_i is an eigenvalue of T, and there is a one-to-one correspondence between eigenvalues of T and the irreducible monic factors of the characteristic polynomial. In general, eigenvalues need not exist, but the irreducible monic factors always exist. In this section, we establish structure theorems based on the irreducible monic factors of the characteristic polynomial instead of eigenvalues.

Definition 7.3.3 Let T be a linear operator on a finite dimensional vector space V with characteristic polynomial

$$f(t) = (-1)^n (\phi_1(t))^{n_1} (\phi_2(t))^{n_2} \cdots (\phi_k(t))^{n_k} \quad \text{where}$$

the $\phi_i(t)$'s, $i = 1, 2, \ldots, k$ are distinct irreducible monic polynomials and n_i's are positive integers. For $i = 1, 2, \ldots, k$ we define the subset K_{ϕ_i} of V by

$$K_{\phi_i} = \{x \in V : (\phi_i(T))^p(x) = \theta \text{ for some positive integer } p\}$$

We show that each K_{ϕ_i} is a non-zero T-invariant subspace of V. Note that if $\phi_i(t) = t-\lambda$, then K_{ϕ_i} is the generalized eigenspace of T corresponding to the eigenvalue λ. Having obtained suitable generalizations of the related concepts of eigenvalues and eigenspace, our next task is to describe a canonical form of a linear operator suitable to this context. The one that we study is called the rational canonical form. Since a canonical form is a description of a matrix representation of a linear operator, it can be defined by specifying of the ordered bases allowed for these representations.

Here the bases of interest naturally arise from the generation of certain cyclic subspaces. For this reason, the student should recall the definition of a T-cyclic subspace generated by a vector and Theorem 2.14.4 (Page-121). We briefly review this concept and introduce some new notation and terminology.

Let T be a linear operator on a finite dimensional vector space V, and let x be a non-zero vector in V. We use the notation C_x for the T-cyclic subspace generated by x. Recall (Theorem 2.14.4) that if $\dim(C_x) = k$, then the set $\{x, T(x), T^2(x), \ldots, T^{k-1}(x)\}$ is an ordered basis for C_x. To distinguish this basis from all other ordered bases for C_x, we call it the T-cyclic basis generated by x and

and denote it by β_x. Let A be the matrix representation of T to C_x relative to the ordered basis β_x. Recall from the proofs of Theorem 2.14.4 that

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & -a_{k-1} \end{pmatrix}$$

where

$$a_0 x + a_1 T(x) + \cdots + a_{k-1} T^{k-1}(x) + T^k(x) = \theta$$

Furthermore, the characteristic polynomial of A is given by

$\det(A - tI) = (-1)^k (a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k)$.

The matrix A is called the <u>companion matrix</u> of the monic polynomial $h(t) = a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k$. Every monic polynomial has a companion matrix and the characteristic polynomial of the companion matrix of a monic polynomial $g(t)$ of degree k is equal to $(-1)^k g(t)$. By Theorem 2.15.8 (page-129), the monic polynomial $h(t)$ is also the minimal polynomial of A. Since A is the matrix representation of the restriction of T to C_x, $h(t)$ is also the minimal polynomial of this restriction. By Theorem 7.3.2, $h(t)$ is also the T-annihilator of x.

It is the objective of this section ~~that~~ to prove that for every linear operator T on a finite dimensional vector space V, there exists an ordered basis β for V such that the matrix representation $[T]_\beta$ is of the form