$$\begin{pmatrix} C_1 & O & \cdots & O \\ O & C_2 & \cdots & O \\ \vdots & \vdots & & \\ O & O & \cdots & C_r \end{pmatrix}$$

where each C_i is the companion matrix of a polynomial $(\phi(t))^m$ such that $\phi(t)$ is a monic irreducible divisor of the characteristic polynomial of T and m is a positive integer. A matrix representation of this kind is called a rational canonical form of T. We call the accompanying basis a rational canonical basis for T.

The next theorem is a simple consequence of the following lemma, which relies on the concept of T-annihilator

Lemma 7.3.4 Let T be a linear operator on a finite dimensional vector space V. Let x be a non-zero vector in V and suppose that the T-annihilator of x is of the form $(\phi(t))^p$ for some irreducible monic polynomial $\phi(t)$. Then $\phi(t)$ divides the minimal polynomial of T, and $x \in K_\phi$.

Proof: By Theorem 7.3.2 (b), $(\phi(t))^p$ divides the minimal polynomial of T. Therefore $\phi(t)$ divides the minimal polynomial of T. Furthermore, $x \in K_\phi$ by the definition of K_ϕ.

Theorem 7.3.5 Let T be a linear operator ~~on~~ on a finite dimensional vector space V, and let β be an ordered basis for V. Then β is a rational canonical basis for T if and only β is the disjoint union of T-cyclic ~~bases~~ bases β_{v_i}, where each v_i, lies in K_ϕ for

for some irreducible monic divisor $\phi(t)$ of the characteristic polynomial of T.

Proof : Exercise.

Example 1 Suppose that T is a linear operator on $\mathbb{R}^8$ and $\beta = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ is a rational canonical basis for T such that

$$C = [T]_\beta = \left(\begin{array}{cc|cccc|cc} 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

is a rational canonical form of T. In this case, the submatrices C_1, C_2 and C_3 are the companion matrices of the polynomials $\phi_1(t)$, $(\phi_2(t))^2$, and $\phi_2(t)$, respectively, where

$$\phi_1(t) = t^2 - t + 3 \quad \text{and} \quad \phi_2(t) = t^2 + 1$$

In the context of Theorem 7.3.5, β is the disjoint union of the T-cyclic bases; that is

$$\beta = \beta_{v_1} \cup \beta_{v_2} \cup \beta_{v_3}$$
$$= \{v_1, v_2\} \cup \{v_3, v_4, v_5, v_6\} \cup \{v_7, v_8\}$$

We can prove that ~~the characteristic~~ (by our previous result) that the characteristic polynomial $f(t)$ of T is the product of the characteristic polynomials of the companion matrices. So, $f(t) = \phi_1(t)(\phi_2(t))^2\phi_2(t) = \phi_1(t)(\phi_2(t))^3$.

The rational canonical form C of the operator T in Example 1 is constructed from the matrices of the form C_i, each of which is the companion matrix of some power of a monic irreducible divisor of the characteristic polynomial of T. Furthermor, each such divisor is used in this way at least once.

In the course of showing that every linear operator T on a finite dimensional vector space has a rational canonical form C, we show that the companion matrices C_i that constitute C are always ~~b~~ constructed from powers of the monic irreducible divisors of the characteristic polynomial of T. A keyrole in our analysis is played by K_ϕ, where $\phi(t)$ is an irreducible monic divisor of the minimal polynomial of T. Since the minimal polynomial of an operator divides the characteristic polynomial of the operator, every irreducible divisor of the former is ~~as~~ also an irreducible divisor of the ~~latter~~ latter. We eventually show that the converse is also true; that is, the minimal polynomial and the characteristic polynomial have the same irreducible divisors.

We begin with a result that lists several properties of the irreducible devisors of the minimal polynomial.

Theorem 7.3.6 Let T be a linear operator on a finite dimensional vector space V, and suppose that

$$p(t) = (\phi_1(t))^{m_1}(\phi_2(t))^{m_2}\cdots(\phi_k(t))^{m_k}$$ is the minimal polynomial of T, where $\phi_i(t)$'s, $i=1,2,\ldots,k$ are distinct irreducible monic factors of $p(t)$ and m_i's are positive integers. Then the following statements are true :

(a) K_{ϕ_i} is a non-zero T-invariant subspace of V for each i.

(b) If x is a non-zero vector in some K_{ϕ_i}, then T-annihilator of x is of the form $(\phi_i(t))^p$ for some integer p.

(c) $K_{\phi_i} \cap K_{\phi_j} = \{\theta\}$ for $i \neq j$

(d) K_{ϕ_i} is invariant under $\phi_j(T)$ for $i \neq j$ and the restriction of $\phi_j(T)$ to K_{ϕ_i} is one-to-one and onto.

(e) $K_{\phi_i} = N\left((\phi_i(T))^{m_i}\right)$ for each i.

Proof: If $k=1$, then (a), (b) and (e) are obvious, while (c) and (c) are vacuously true. Now suppose that $k>1$.

(a) The proof ~~of~~ ~~@~~ that K_{ϕ_i} is a T-invariant subspace is left as an exercise. Let $f_i(t)$ be the polynomial obtained from $p(t)$ by omitting the factor $(\phi_i(t))^{m_i}$. To prove that K_{ϕ_i} is non-zero, first observe that $f_i(t)$ is a proper divisor of $p(t)$; therefore there exists a vector $z \in V$ such that ~~z~~ $x = f_i(T)(z) \neq \theta$. Then $x \in K_{\phi_i}$, because $(\phi_i(T))^{m_i}(x) = (\phi_i(T))^{m_i} f_i(T)(z) = p(T)(z) = \theta$

(b) Assume the hypothesis. Then $(\phi_i(T))^q(x) = \theta$ for some positive integer q. Hence the T-annihilator of x divides $(\phi_i(t))^q$ by Theorem 7.3.2(b) and the result follows.

(c) Assume $i \neq j$, Let $x \in K_{\phi_i} \cap K_{\phi_j}$ and suppose that $x \neq \theta$. By (b), the T-annihilator ~~is~~ of x is a power of both $\phi_i(t)$ and $\phi_j(t)$. But this is impossible because $\phi_i(t)$ and $\phi_j(t)$ are relatively