prime. So, we conclude that $x = \theta$.

(d) Assume $i \neq j$. Since K_{ϕ_i} is T-invariant, it is also $\phi_j(T)$-invariant. Suppose that $\phi_j(T)(x) = \theta$ for some $x \in K_{\phi_i}$. Then $x \in K_{\phi_i} \cap K_{\phi_j} = \{\theta\}$ by (c). Therefore the restriction of $\phi_j(T)$ to K_{ϕ_i} is one-to-one. Since V is finite dimentional, this restriction is also onto.

(e) Suppose i be one of $1, 2, \ldots, k$. Clearly, $N\left((\phi_i(T))^{m_i}\right) \subseteq K_{\phi_i}$. Let $f_i(t)$ be the polynomial defined in (a). Since $f_i(t)$ is a product of polynomials of the form $\phi_j(t)$, $j \neq i$, we have by (d) that the restriction of $f_i(T)$ to K_{ϕ_i} is onto. Let $x \in K_{\phi_i}$. Then there exists $y \in K_{\phi_i}$ such that $f_i(T)(y) = x$. So,

$$\left((\phi_i(T))^{m_i}\right)(x) = \left((\phi_i(T))^{m_i}\right) f_i(T)(y) = p(T)(y) = \theta$$

and hence $x \in N\left((\phi_i(T))^{m_i}\right)$.

Thus $K_{\phi_i} = N\left((\phi_i(T))^{m_i}\right)$

Since a rational canonical basis for an operator T is obtained from a union of T-cyclic bases, we need to know when such a union is linearly independent. The next major result, Theorem 7.3.8 reduces this problem to the study of T-cyclic bases within K_ϕ, where $\phi(t)$ is an irreducible monic divisor of the minimal polynomial of T. We begin with the following lemma:

Lemma 7.3.7 Let T be a linear operator on a finite dimensional vector space V, and suppose that

$p(t) = (\phi_1(t))^{m_1}(\phi_2(t))^{m_2}\cdots(\phi_k(t))^{m_k}$ is the minimal polynomial of T, where the ϕ_i's, $i=1,2,\ldots,k$ are distinct irreducible monic factors of $p(t)$ and the m_i's are positive integers. For $i=1,2,\ldots,k$, let $v_i \in K_{\phi_i}$ be such that

$$v_1+v_2+\cdots+v_k=\theta \qquad \cdots\cdots (1)$$

Then $v_i=\theta$ for all i.

Proof: The result is trivial for $k=1$, suppose that $k>1$. Consider any i, let $f_i(t)$ be the polynomial obtained from $p(t)$ by omitting the factor $(\phi_i(t))^{m_i}$. As a consequence of Theorem 7.3.6, $f_i(T)$ is one-to-one on K_{ϕ_i}, and $f_i(T)(v_j)=\theta$ for $i\neq j$. Then applying $f_i(T)$ to (1), we obtain $f_i(T)(v_i)=\theta$ from which it follows that $v_i=\theta$.

Theorem 7.3.8 Let T be a linear operator on a finite dimensional vector space V, and suppose that

$p(t) = (\phi_1(t))^{m_1}(\phi_2(t))^{m_2}\cdots(\phi_k(t))^{m_k}$ is the minimal polynomial of T, where the ϕ_i's, $i=1,2,\ldots,k$ are the distinct irreducible monic factors of $p(t)$ and the m_i's are positive integers. For $i=1,2,\ldots,k$, let S_i be a linearly independent subset of K_{ϕ_i}. Then

(a) $S_i \cap S_j = \phi$ for $i\neq j$

(b) $S_1\cup S_2\cup\cdots\cup S_k$ is linearly independent.

Proof: If $k_0 = 1$ then (a) is vacuously true and (a) is obvious. Now suppose that $k > 1$. Then (a) follows immediately from Theorem 7.3.6 (c). Furthermore, the proof of (b) is identical to the proof of Theorem 4.13.7 (page – 109) with the eigenspaces replaced by the subspaces K_{ϕ_i}.

In view of Theorem 7.3.8, we can focus on bases of individual spaces of the form $K_{\phi(t)}$, where $\phi(t)$ is an irreducible monic divisor of the minimal polynomial of T. The next several results gives us ways to construct bases for these spaces that are union of T-cyclic bases. These results serve the dual purposes of leading to the existence theorem for the rational canonical form and of providing methods for constructing rational canonical bases.

For Theorems 7.3.9 and ~~7.3.9~~ 7.3.11 and the latter's corollary, we fix a linear operator T on a finite dimensional vector space V and an irreducible monic divisor $\phi(t)$ of the minimal polynomial of T.

Theorem 7.3.9 Let $v_1, v_2, \ldots, v_k$ be distinct vectors in K_ϕ such that $S_1 = \beta_{v_1} \cup \beta_{v_2} \cup \cdots \cup \beta_{v_k}$ is linearly independent. For each i, choose $w_i \in V$ such that $\phi(T)(w_i) = v_i$. Then $S_2 = \beta_{w_1} \cup \beta_{w_2} \cup \cdots \cup \beta_{w_k}$ is also linearly independent.

Proof: Consider any linear combination of vectors in S_2 that sums to zero, say

$$\sum_{i=1}^{k} \sum_{j=0}^{n_i} a_{ij} T^j(w_i) = \theta \qquad \text{—(1)}$$

For each i, let $f_i(t)$ be the polynomial defined by

$$f_i(t) = \sum_{j=0}^{n_i} a_{ij} t^j$$

Then (1) can be written as

$$\sum_{i=1}^{k} f_i(T)(w_i) = \theta \quad --- (2)$$

Apply $\phi(T)$ both sides of (2) to obtain

$$\sum_{i=1}^{k} \phi(T) f_i(T)(w_i) = \sum_{i=1}^{k} f_i(T)\,\phi(T)(w_i) = \sum_{i=1}^{k} f_i(T)(v_i) = \theta$$

The last sum can be ~~written~~ rewritten as a ~~a~~ linear combination of the vectors in S_1 so that each $f_i(T)(v_i)$ is a linear combination of the vectors in β_{v_i}. Since S_1 is linearly, it follows that $f_i(T)(v_i) = \theta$ for all i.

So, by Theorem 7.3.2, the T-annihilator of v_i divides $f_i(t)$ for all i. By Theorem 7.3.6(b), $\phi(t)$ divides the T-annihilator of v_i and hence $\phi(t)$ divides $f_i(t)$ for all i. Thus for each i, there exists a polynomial $g_i(t)$ such that $f_i(t) = g_i(t)\,\phi(t)$.

So, (2) becomes

$$\sum_{i=1}^{k} g_i(T)\phi(T)(w_i) = \sum_{i=1}^{k} g_i(T)(v_i) = \theta$$

Again linear independence of S_1, requires that

$$f_i(T)(w_i) = g_i(T)(v_i) = \theta \text{ for all } i.$$

But $f_i(T)(w_i)$ is the result of grouping the terms of linear combination in (1) that arise from the linearly independent set β_{w_i}, we conclude that