has a rational canonical basis β_1 consisting of a union of U-cyclic bases (and hence T-cyclic bases) of vectors from some of the subspaces K_{ϕ_i}, $1 \le i \le k-1$. By the Corollary 7.3.12 of Theorem 7.3.11, K_{ϕ_k} has a basis β_2 consisting of a union of T-cyclic bases. By theorem 7.3.8, β_1 and β_2 are disjoint, and $\beta = \beta_1 \cup \beta_2$ is linearly independent. Let s denotes the number of vectors in β.

Then $s = \dim\left(R\left(\phi_k(T)\right)^{m_k}\right) + \dim\left(K_{\phi_k}\right)$

$$= \text{rank}\left(\left(\phi_k(T)\right)^{m_k}\right) + \text{nullity}\left(\left(\phi_k(T)\right)^{m_k}\right)$$

$$= n$$

We conclude that β is a basis of V. Therefore β is a rational canonical basis and T has a rational canonical form.

In our study of the rational canonical form, we relied on the minimal polynomial. We are now able to relate the rational canonical form to the characteristic polynomial in this theorem that follows and state it without proof.

<u>Theorem 7.3.14</u> Let T be a linear operator on a n-dimensional vector space V with characteristic polynomial

$$f(t) = (-1)^n\left(\phi_1(t)\right)^{n_1}\left(\phi_2(t)\right)^{n_2}\cdots\left(\phi_k(t)\right)^{n_k}$$ where the ϕ_i's, $i=1,2,\ldots,k$ are distinct irreducible monic polynomials and the n_i's are positive integers. Then the following statements are true:

(a) $\phi_1(t), \phi_2(t), \ldots, \phi_k(t)$ are the irreducible monic factors of the minimal polynomial.

(b) For each i, $\dim(K_{\phi_i}) = d_i n_i$ where d_i is the degree of $\phi_i(t)$

(c) If β is a ~~ra~~ rational canonical basis for T then $\beta_i = \beta \cap K_{\phi_i}$ is a basis for K_{ϕ_i} for each i.

(d) If γ_i is a basis for K_{ϕ_i} for each i, then $\gamma = \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_k$ is a basis for V. In particular, if each γ_i is a disjoint union ~~of~~ of T-cyclic bases, then γ is a rational canonical basis for T.

Uniqueness of the rational Canonical form

Having shown that a rational canonical form exists, we are now in a position to ask about the extent to which it is unique. Certainly, the rational canonical form of a linear operator T can be modified by permuting the T-cyclic bases that constitute that ~~correspond~~ constitute the corresponding rational canonical ~~form~~ basis. This has the effect of permuting the companion matrices that make up the rational canonical form. As in the case of Jordan Canonical form, we show that except for these permutations, the rational Canonical form is unique, although the rational ~~c~~ Canonical bases are not.

To simplify this task, we adopt the convention of ordering every rational canonical basis so that all the T-cyclic bases associated with same irreducible monic divisor of the characteristic polynomial are grouped together. Furthermore, within each grouping, we arrange the T-cyclic bases ~~in~~ in decreasing order of ~~sizes~~ size. Our task is to show that, subject to this order, the rational canonical form of a linear operator is unique upto the arrangement of the arrangement of the irreducible monic divisors.

As in the case of Jordan canonical form, we introduce arrays of dots from which we can reconstruct the rational canonical form. For the Jordan Canonical form, we devised a dot diagram for each eigenvalue of the given operator. In the case of rational canonical form, we define a dot diagram for each irreducible monic divisor of the characteristic polynomial of the given operator. A proof that the resulting dot diagrams are completely determined by the operator is also a proof that the rational canonical form is unique.

In what follows, T is a linear operator on a finite dimensional vector space with rational canonical basis β, $\phi(t)$ is an irreducible monic divisor of the characteristic polynomial of T; $\beta_{v_1}, \beta_{v_2}, \ldots, \beta_{v_k}$ are the T-cyclic bases of β that are contained in K_ϕ; and d is the degree of $\phi(t)$. For each j, let $(\phi(t))^{p_j}$ be the annihilator of v_j. This polynomial has degree dp_j; therefore by Theorem 7.3.2, β_{v_j} contains dp_j vectors. Furthermore, $p_1 \geq p_2 \geq \cdots \geq p_k$ since the T-cyclic bases are arranged in decreasing order of size. We define the dot diagram of $\phi(t)$ to be the array consisting of k columns of dots with p_j dots in the jth column, arranged so that jth begins at the top and terminates after p_j dots.

For example, if $k=3$, $p_1=4$, $p_2=2$ and $p_3=2$, then the dot diagram is

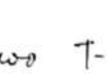
$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & & \\ \bullet & & \end{array}$$

Although each column of a dot diagrams corresponds to a

T-cyclic basis β_{v_i} in K_ϕ, there are fewer dots in the column than there vectors in the basis.

Example 2 Recall the linear operator T of Example 1 (Page-174) with the the rational canonical basis β and the rational canonical form $C=[T]_\beta$. Since there are two irreducible monic divisors of the characteristic polynomials of T, $\phi_1(t)=t^2-t+3$, and $\phi_2(t)=t^2+1$, there are two dot diagrams to consider. Because $\phi_1(t)$ is the T-annihilator of v_1 and β_{v_1} is a basis for K_{ϕ_1} and the dot diagram for $\phi_1(t)$ consists of a single dot. The other two T-cyclic bases β_{v_3} and β_{v_7}, lie in K_{ϕ_2}. Since v_3 has a T-annihilator $(\phi_2(t))^2$ and v_7 has T-annihilator $\phi_2(t)$, so in the dot diagram of $\phi_2(t)$ we have $p_1=2$ and $p_2=1$. These diagrams are as follows:

Dot diagram for $\phi_1(t)$



Dot diagram for $\phi_2(t)$

In practice, we obtain the rational canonical form of a linear operator from the information provided by dot diagrams. This is illustrated in the next example.

Example 3 Let T be a linear operator on a finite dimensional vector space over $\mathbb{R}$, and suppose that the irreducible monic divisors of the characteristic polynomial of T are

$$\phi_1(t)=t-1,\quad \phi_2(t)=t^2+2 \quad \text{and} \quad \phi_3(t)=t^2+t+1$$

Suppose, furthermore that the dot diagrams associated with these divisors are as follows:

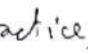


Diagram for $\phi_1(t)$

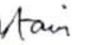


Diagram for $\phi_2(t)$

Diagram for $\phi_3(t)$