Since the dot diagrams for $\phi_1(t)$ has two ~~dots~~ columns, it contributes two companion matrices to the rational canonical form. The first column has two dots, and therefore corresponds to the 2×2 companion matrix of $(\phi_1(t))^2 = (t-1)^2$. The second column, with only one dot, corresponds to the 1×1 companion matrix of $\phi_1(t) = t-1$. These two companion matrices are given by $C_1 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ and $C_2 = (1)$.

The dot diagram for $\phi_2(t) = t^2+2$ consists of two columns, each containing a single dot; hence ~~the~~ this diagram contributes two copies of 2×2 companion matrix for $\phi_2(t)$ namely,

$$C_3 = C_4 = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$$

The dot diagram for $\phi_3(t) = t^2+t+1$ consists of a single column with a single dot contributing the single 2×2 companion matrix $C_5 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$

Therefore the rational canonical form of T is the 9×9 matrix

$$C = \begin{pmatrix} C_1 & O & O & O & O \\ O & C_2 & O & O & O \\ O & O & C_3 & O & O \\ O & O & O & C_4 & O \\ O & O & O & O & C_5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

we return to the general problem of finding dot diagrams. As we did before, we fix a linear operator T on a finite dimensional vector space and an irreducible monic divisor $\phi(t)$ of the characteristic polynomial of T. Let U denote the restriction of the linear operator $\phi(T)$ to K_ϕ. By Theorem 7.3.6(d), $U^q = T_0$ for some positive integer q. Consequently, the characteristic polynomial of U is $(-1)^m t^m$, where $m = \dim(K_\phi)$. Therefore K_ϕ is the generalised eigenspace of U corresponding to $\lambda = 0$, and U has a Jordan canonical form. The dot diagram associated with the Jordan canonical form of U gives us a key to understanding the dot diagram of T that is associated with $\phi(t)$. We now relate the two diagrams.

Let β be a rational canonical basis for T and $\beta_{v_1}, \beta_{v_2}, \ldots, \beta_{v_k}$ be the T-cyclic bases of β that are contained in K_ϕ. Consider one of these T-cyclic bases β_{v_j}, and suppose again that the T-annihilator v_j is $(\phi(t))^{p_j}$. Then β_{v_j} consists of dp_j vectors in β. For $0 \le i < d$, let γ_i be the cycle of generalized eigenvectors of U corresponding to $\lambda = 0$ with end vector $T^i(v_j)$, where $T^0(v_j) = b_j$. Then

$$\gamma_i = \left\{ (\phi(T))^{p_j-1} T^i(v_j), (\phi(T))^{p_j-2} T^i(v_j), \cdots (\phi(T)) T^i(v_j), T^i(v_j) \right\}.$$

By Theorem 7.1.3 (Page-138), γ_i is a linearly independent subset of C_{v_i}. Now let

$$\alpha_j = \gamma_0 \cup \gamma_1 \cup \cdots \cup \gamma_{d-1}.$$

Notice that α_j contains $p_j d$ vectors.

Lemma 7.3.15 α_j is an ordered basis for C_{v_j}

Proof: The key to this proof is Theorem 7.1.6 (Page-142). Since α_j is the union of cycles of generalized eigen-vectors of U corresponding to $\lambda=0$. It suffices to show that the set of initial vectors of these cycles

$$\{(\phi(T))^{p_j-1}(v_j), (\phi(T))^{p_j-1}T(v_j), \cdots, (\phi(T))^{p_j-1}T^{d-1}(v_j)\}$$

is linearly independent. Consider any linear combination of these vectors

$$a_0(\phi(T))^{p_j-1}(v_j) + a_1(\phi(T))^{p_j-1}T(v_j) + \cdots + a_{d-1}(\phi(T))^{p_j-1}T^{d-1}(v_j),$$

where not all of the coefficients are zero. Let $g(t)$ be the polynomial defined by $g(t) = a_0 + a_1 t + \cdots + a_{d-1}t^{d-1}$. Then $g(t)$ is a non-zero polynomial of degree less than d, and hence $\{\phi(t)\}^{p_j-1}g(t)$ is a non-zero polynomial with degree less than $p_j d$. Since $(\phi(t))^{p_j}$ is the T-annihilator of v_j, it follows that $(\phi(T))^{p_j-1}g(T)(v_j) \neq \theta$. Therefore the set of initial vectors is linearly independent. So, by Theorem 7.1.6, α_j is linearly independent and the γ_i's are disjoint. Consequently, α_j contains $p_j d$ linearly independent vectors in C_{v_j}, which has dimension $p_j d$. We conclude that α_j is a basis for C_{v_j}.

Thus we may replace β_{v_j} by α_j as a basis for C_{v_j}. We do this for each j to obtain a subset

$$\alpha = \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_k \text{ of } K_\phi.$$

Lemma 7.3.16 α is a Jordan canonical basis for K_ϕ.

Proof: Since $\beta_{v_1} \cup \beta_{v_2} \cup \cdots \cup \beta_{v_k}$ is a basis for K_ϕ and since $\text{span}(\alpha_i) = \text{span}(\beta_{v_i}) = C_{v_i}$. This implies that α is a basis for K_ϕ. Because α is a union of cycles of generalized eigenvectors of U, we conclude that α is a Jordan canonical basis.

We are now in a position to relate the dot diagram of T corresponding to $\phi(t)$ to the dot diagram of U, bearing in mind that in the first case we are considering a rational canonical form and in the second case we are considering a Jordan canonical form. For convenience, we designate the first diagram D_1 and the second diagram D_2. For each j, the presence of the T-cyclic basis β_{x_j} results in a column of p_j dots in D_1. By Lemma 7.3.15, this basis is replaced by the union α_j of d cycles of generalized eigenvectors of U, each of length p_j, which becomes part of the Jordan canonical basis for U. In effect, α_j determines d columns each containing p_j dots in D_2. So each column in D_1 determines d columns in D_2 of same length, ~~at~~ and all columns of D_2 are obtained in this way. Alternatively, each row in D_2 has d times as many dots as the corresponding row in D_1. Since Theorem 7.2.3 (page-159) gives us the number of dots in any row of D_2, we may divide the appropriate expression in this theorem by d to obtain the number of dots