in the corresponding rows of D_1. Thus we have the following result:

Theorem 7.3.17 Let T be a linear operator on a finite dimensional vector space V, let $\phi(t)$ be an irreducible monic divisor of the characteristic polynomial of T of degree d, and let r_i denote the number of dots in the ith row of the dot diagram for $\phi(t)$ with respect to a rational canonical basis for T.

Then (a) $r_1 = \frac{1}{d}\left[\dim(V) - \operatorname{rank}(\phi(T))\right]$

(b) $r_i = \frac{1}{d}\left[\operatorname{rank}\left((\phi(T))^{i-1}\right) - \operatorname{rank}\left((\phi(T))^{i}\right)\right]$, for $i > 1$

Thus the dot diagram associated with a rational canonical form of an operator are completely determined by the operator. Since the rational canonical form is completely determined by its dot diagrams, we have the following uniqueness condition:

Corollary 7.3.18 Under the condition described earlier, the rational canonical form of a linear operator is unique up to the arrangement of the irreducible monic divisors of the characteristic polynomial.

Since the rational canonical form of a linear operator is unique, the polynomials corresponding to the companion matrices that determine this form are also unique. These polynomials, which are powers of the irreducible monic divisors, are called the elementary divisors of the linear operator. Since a companion matrix may occur more than once in a rational canonical form, the same is true for elementary divisors. We call the number of such occurrences the multiplicity of the elementary divisors.

Conversely, the elementary divisors and their multiplicities determine the companion matrices and, therefore, the rational canonical form of a linear operator.

Example 4

Let $\beta = \{e^x\cos 2x, e^x\sin 2x, xe^x\cos 2x, xe^x\sin 2x\}$ be viewed as a subset of $F(\mathbb{R}, \mathbb{R})$ the space of all real valued functions defined on $\mathbb{R}$ and let $V = \text{span}(\beta)$. Then V is a four dimensional subspace of $F(\mathbb{R}, \mathbb{R})$, and β is an ordered basis for V. Let D be the linear operator on V defined by $D(y) = y'$, the derivative of y, and let $A = [D]_\beta$. Then

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

and the characteristic polynomial of D, and hence of A, is $f(t) = (t^2 - 2t + 5)^2$

Thus $\phi(t) = t^2 - 2t + 5$ is the only irreducible monic divisor of $f(t)$. Since $\phi(t)$ has degree 2, and V is four dimensional, the dot diagram for $\phi(t)$ contains only two dots. Therefore the dot diagram is determined by r_1, the number of dots in the first row. Because ranks are preserved under matrix representations, we can use A in place of D in the formula given in Theorem 7.3.17. Now

$$\phi(A) = \begin{pmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and so

$$r_1 = \frac{1}{2}\left[4 - \text{rank}\left(\phi(A)\right)\right] = \frac{1}{2}(4-2) = 1$$

It follows that the second dot lies in the second row, and the dot diagram is as follows:

$$\begin{array}{c} \bullet \\ \bullet \end{array}$$

Hence V is a D-cyclic space generated by a single function with D-annihilator $(\phi(t))^2$. Furthermore its rational canonical form is given by the companion matrix of $(\phi(t))^2 = t^4 - 4t^3 + 14t^2 - 20t + 25$, which is

$$\begin{pmatrix} 0 & 0 & 0 & -25 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

Thus $(\phi(t))^2$ is the only elementary divisor of D, and it has multiplicity 1. For the cyclic generator, it suffices to find a function g in V for which $\phi(D)(g) \neq 0$. Since $\phi(A)(e_3) \neq 0$, it follows that $\phi(D)(xe^x \cos 2x) \neq 0$; Therefore $g(x) = xe^x \cos 2x$

can be chosen as the cyclic generator. Hence

$\beta_g = \{xe^x\cos 2x, D(xe^x\cos 2x), D^2(xe^x\cos 2x), D^3(xe^x\cos 2x)\}$

is a rational canonical basis for D. Notice that the function h defined by $h(x) = xe^x\sin 2x$ can be chosen in place of g. This shows that the rational canonical basis is not unique.

It is convenient to refer to the rational canonical form and elementary divisors of a matrix, which are defined in the obvious way.

Definition 7.3.19 Let $A \in M_{n\times n}(F)$. The rational canonical form of A is defined to ~~be~~ be the rational canonical form of L_A. Likewise, for A, the elementary divisors and their multiplicities are the same as ~~that~~ those of L_A.

Let A be an $n\times n$ matrix, let C be a rational canonical form of A, and let β be the appropriate canonical basis for L_A. Then $C = [L_A]_\beta$, and therefore A is similar to C. In fact, if Q is the matrix whose columns are the vectors of β in the same order, then $Q^{-1}AQ = C$.

Example 5 For the following real matrix A, we find the rational canonical form C of A and a matrix Q such that $Q^{-1}AQ = C$,

$$A = \begin{pmatrix} 0 & 2 & 0 & -6 & 2 \\ 1 & -2 & 0 & 0 & 2 \\ 1 & 0 & 1 & -3 & 2 \\ 1 & -2 & 1 & -1 & 2 \\ 1 & -4 & 3 & -3 & 4 \end{pmatrix}$$