The characteristic polynomial of A is $f(t) = -(t^2+2)^2(t-2)$; therefore $\phi_1(t) = t^2+2$, $\phi_2(t) = t-2$ are the distinct irreducible monic divisors of $f(t)$. By Theorem 7.3.14, $\dim(K_{\phi_1}) = 4$, and $\dim(K_{\phi_2}) = 1$. Since the degree of $\phi_1(t)$ is 2, the total number of dots in the dot diagram of $\phi_1(t) = 4/2 = 2$ and the number of dots r_1 in the first row is given by

$$r_1 = \frac{1}{2}\left[\dim(\mathbb{R}^5) - \operatorname{rank}(\phi_1(A))\right]$$

$$= \frac{1}{2}\left[5 - \operatorname{rank}(A^2+2I)\right]$$

$$= \frac{1}{2}\left[5-1\right] = 2$$

Thus the dot diagram is

$$\bullet \quad \bullet$$

and each column contributes the companion matrix

$$\begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$$

for $\phi_1(t) = t^2+2$ to the rational canonical form C. Consequently $\phi_1(t)$ is an elementary divisors with multiplicity 2. Since $\dim(K_{\phi_2}) = 1$, the dot diagram of $\phi_2(t) = t-2$ consists of a single dot, which contributes the 1×1 matrix (2). Hence $\phi_2(t)$ is an elementary divisor with multiplicity 1. Therefore the rational canonical form C is given by

$$C = \left(\begin{array}{cc|cc|c} 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 \end{array}\right)$$

We can infer from the dot diagram of $\phi_1(t)$ that if β is a rational canonical basis for L_A, then $\beta \cap K_{\phi_1}$ is the union of two ~~cycles~~ cyclic bases β_{v_1} and β_{v_2} where v_1 and v_2 ~~are~~ each have annihilator ~~$\phi_1(t)$~~ $\phi_1(t)$. It follows that both v_1 and v_2 lie in $N(\phi_1(L_A))$. It can be shown that

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $N(\phi_1(L_A))$. Setting $v_1 = e_1$, we see that $Av_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Next choose v_2 in $K_{\phi_1} = N(\phi(L_A))$, but not in the span of $\beta_{v_1} = \{v_1, Av_1\}$. For example, $v_2 = e_2$. Then it can be ~~shown~~ seen that

$$Av_2 = \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \\ -4 \end{pmatrix}$$

and $\beta_{v_1} \cup \beta_{v_2}$ is a basis for K_{ϕ_1}.

Since the dot diagram of $\phi_2(t) = t-2$ consists of a single dot, any non-zero vector of K_{ϕ_2} is an eigenvector of A corresponding to the eigenvalue $\lambda = 2$. For example, choose

$$v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

By Theorem 7.3.14, $\beta = \{v_1, Av_1, v_2, Av_2, v_3\}$ is a rational canonical basis for L_A. So, setting

$$Q = \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & -4 & 2 \end{pmatrix}$$

we have $Q^{-1}AQ = C$

Example 6

For the following matrix A, we find the rational canonical form C and a matrix Q such that $Q^{-1}AQ = C$:

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Since the characteristic polynomial of A is $f(t) = (t-2)^4$, the only irreducible monic divisor of $f(t)$ is $\phi(t) = t-2$ and so $K_\phi = \mathbb{R}^4$. In this case $\phi(t)$ has degree 1; hence in applying Theorem 7.3.17 to compute the dot diagram for $\phi(t)$, we obtain

$$r_1 = 4 - \operatorname{rank}(\phi(A)) = 4-2 = 2$$

$$r_2 = \operatorname{rank}(\phi(A)) - \operatorname{rank}((\phi(A))^2) = 2-1 = 1$$

and $$r_3 = \operatorname{rank}((\phi(A))^2) - \operatorname{rank}((\phi(A))^3) = 1-0 = 1$$

where r_i is the number of dots in the ith row of the dot diagram. Since there are $\dim(\mathbb{R}^4) = 4$ dots in the diagram, we may terminate these computations with r_3. Thus the dot diagram for A is $\begin{array}{ll} \bullet & \bullet \\ \bullet & \\ \bullet & \end{array}$

Since $(t-2)^3$ has the companion matrix $\begin{pmatrix} 0 & 0 & 8 \\ 1 & 0 & -12 \\ 0 & 1 & 6 \end{pmatrix}$

and $(t-2)$ has the companion matrix (2), the rational canonical form of A is given by

$$C = \left(\begin{array}{ccc|c} 0 & 0 & 8 & 0 \\ 1 & 0 & -12 & 0 \\ 0 & 1 & 6 & 0 \\ \hline 0 & 0 & 0 & 2 \end{array}\right)$$

Now we find a rational canonical basis for L_A. The preceding dot diagram indicates that there are two vectors v_1 and v_2 in $\mathbb{R}^4$ with annihilators $(\phi(t))^3$ and $\phi(t)$ respectively and such that

$\beta = \{\beta_{v_1} \cup \beta_{v_2}\} = \{v_1, Av_1, v_2, Av_2\}$ is a rational canonical basis for L_A. Furthermore $v_1 \notin N((L_A-2I)^2)$ and $v_2 \in N((L_A-2I))$. It can be easily shown that $N(L_A-2I) = \operatorname{span}(\{e_1, e_4\})$ and $N((L_A-2I)^2) = \operatorname{span}(\{e_1, e_2, e_4\})$. The standard vector e_3 meets the criteria for v_1; so we set $v_1 = e_3$. It follows that $Av_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$ and $A^2v_1 = \begin{pmatrix} 1 \\ 4 \\ 4 \\ 0 \end{pmatrix}$. Next we choose a vector $v_2 \in N(L_A-2I)$ that is not in the span of β_{v_1}. Clearly $v_2 = e_4$ satisfies this condition. Thus $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a rational canonical basis for L_A. Finally let Q be the matrix whose columns are vectors of β in the same order. So $Q = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Then $C = Q^{-1}AQ$ THIS IS THE END OF CC12 NOTES