$\sigma(a) = a$ or a^2, $\sigma(b) = b$, ab or a^2b (as $o(\sigma(x)) = o(x)$).
Moreover, when $\sigma(a)$ and $\sigma(b)$ are fixed, $\sigma(x)$ is known for every $x \in S_3$. Hence σ is completely determined. Thus there cannot be more than six automorphisms of S_3. Hence $Aut(S) \simeq Inn(S) \simeq S_3$

3. Let G be a finite abelian group of order n and let m be a positive integer less than n and prime to n. Then show that the mapping $\sigma: G \to G$ defined by $\sigma(x) = x^m$, $x \in G$ is an automorphism.

Solution: Since $\gcd(m,n) = 1$. So $\exists$ integers u and v such that $mu + nv = 1$. We show that σ is onto. Let $x \in G$

Then $x = x^1 = x^{mu+nv} = x^{mu} x^{nv} = x^{mu}$, since $o(G) = n$

So, $(x^u)^m = x$ or, $\sigma(x^u) = x$. So, σ is onto.

We show now that σ is injective. We show that $Ker\,\sigma = \{e\}$, e is the identity element in G. Let $x \in Ker\,\sigma$

then $\sigma(x) = e$ or, $x^m = e$. As $\gcd(m,n) = 1$

$\exists$ integers u and v such that $mu + nv = 1$. Now

$x = x^{mu+nv} = x^{mu} \cdot x^{nv} = x^{mu}$ as $o(G) = n$

Now $x^m = e \Rightarrow x^{mu} = e \Rightarrow x = e$ So, $Ker\,\sigma = \{e\}$

So, σ is injective. Now let $x, y \in G$

Then $\sigma(xy) = (xy)^m = x^m y^m$ (as G is abelian)

$= \sigma(x)\sigma(y)$

So, σ is a homomorphism. Hence σ is an automorphism.

4. Prove that a finite group G having more than two elements and with the condition that $x^2 \neq e$ for some $x \in G$ must have a non-trivial automorphism

Solution: When G is abelian, the mapping $\phi: G \to G$ defined by $\phi(a) = a^{-1}$, $a \in G$ is an automorphism, and, clearly, ϕ is not an identity automorphism as there exists some $x \in G$ such that $x^2 \neq e$ or $x \neq x^{-1}$. When G is ~~abelian~~ not abelian there exists a non-trivial inner automorphism (by Corollary 1.1.8)

5. (i) Let G be a group and $f: G \to G$ be defined by $f(a) = a^n$ for all $a \in G$, where n is a positive integer. Suppose f is an automorphism. Prove that $a^{n-1} \in Z(G)$ for all $a \in G$.

(ii) Let G be a group and $f: G \to G$ defined by $f(a) = a^3$ for all $a \in G$, is an automorphism. Prove that G is commutative.

Solution: (i) Let $a, b \in G$ Then $f(a^{-1}ba) = (a^{-1}ba)^n = a^{-1}b^n a$.

So, $a^{-n}b^n a^n = f(a^{-1})f(b)f(a) = f(a^{-1}ba) = a^{-1}b^n a$.

Hence, $a^{-(n-1)}b^n a^{n-1} = b^n$ or, $\left(a^{-(n-1)} b\, a^{n-1}\right)^n = b^n$

So, $f\left(a^{-(n-1)} b\, a^{n-1}\right) = f(b)$. Since f is injective,

$a^{-(n-1)} b\, a^{n-1} = b$. Hence $a^{n-1} b = b a^{n-1}$, proving that $a^{n-1} \in Z(G)$.

(ii) By (i) $a^2 \in Z(G)$ for all $a \in G$. Let $a, b \in G$. Then

$$f(ab) = (ab)^3 = ab(ab)^2 = a(ab)^2 b \quad (\text{As } (ab)^2 \in Z(G))$$
$$= aabab\,b = a^2bab^2 = ba^2b^2a \quad [\text{As } a^2 \in Z(G) \text{ and } b^2 \in Z(G)]$$
$$= bb^2a^2a \quad [\text{As } b^2 \in Z(G)]$$
$$= b^3a^3$$
$$= f(b)f(a) = f(ba) \quad [\text{As } f \text{ is a homomorphism}]$$

Hence $ab = ba$ since f is injective. So, G is commutative.

6. Let G be a finite group and f be an automorphism of G such that for all $a \in G$, $f(a) = a$ if and only if $a = e$ (e is the identity element in G). Show that for all $g \in G$, there exists $a \in G$ such that $g = a^{-1} f(a)$

Solution: Let $G = \{a_1, a_2, \ldots, a_n\}$. Let $S = \{a_1^{-1} f(a_1), a_2^{-1} f(a_2), \ldots, a_n^{-1} f(a_n)\}$. Then $S \subseteq G$. Next, we show that all elements of S are distinct.

Now $a_i^{-1} f(a_i) = a_j^{-1} f(a_j)$ if and only if $f(a_i)\left(f(a_j)\right)^{-1} = a_i a_j^{-1}$ if and only if $f(a_i) f(a_j^{-1}) = a_i a_j^{-1}$ if and only if $f(a_i a_j^{-1}) = a_i a_j^{-1}$ if and only if $a_i a_j^{-1} = e$ if and only if $a_i = a_j$. This shows that all elements of S are distinct and so $o(S) = n$. Thus $S = G$.

Let $g \in G$. Then $g \in S$. Hence, $g = a^{-1} f(a)$ for some $a \in G$.

7. Let G be a finite group and f be an automorphism of G, such that ~~$f(a) = a$~~ for all $a \in G$, $f(a) = a$ if and only if $a = e$. Suppose $f^2 = I_G$, where I_G denotes the identity map. Prove that G is commutative.

Solution: Let $g \in G$. By previous exercise (6.), $g = a^{-1} f(a)$ for some $a \in G$. Then $g = I_G(g) = f^2(a^{-1} f(a)) = f(f(a^{-1} f(a)))$ $= f(f(a^{-1}) f^2(a)) = f((f(a))^{-1} a) = f(g^{-1})$. This implies $f(g) = g^{-1}$ (as $f^2 = I_G$) for all $g \in G$. Let $a, b \in G$.

Then $(ab)^{-1} = f(ab) = f(a) f(b) = a^{-1} b^{-1} = (ba)^{-1}$ and so $ab = ba$. Hence, G is commutative.

8. Let G be a finite abelian group of order n where n is odd and $n > 1$. Show that G has a non-trivial automorphism

Solution: Exercise

Q Let $f: G \to G$ be a homomorphism (i.e., f is an endomorphism of G). Suppose f commutes with every inner automorphism of G. Show that

(i) $K = \{x \in G : f^2(x) = f(x)\}$ is a normal subgroup of G.

(ii) G/K is abelian

Solution: $f^2(e) = f(f(e)) = f(e) \Rightarrow e \in K$ (where e is the identity element in G). So, K is non-empty. Let $x, y \in K$

Then $f(x) = f^2(x)$ and $f(y) = f^2(y)$

So, $f^2(xy^{-1}) = f(f(xy^{-1})) = f(f(x) f(y^{-1}))$

$= \cancel{f(f(x)(f(y))^{-1}) = f^2(x)}$

$= \cancel{f^2(x) f^2(y^{-1})}\ f(f(x)(f(y))^{-1})$

$= f^2(x) f((f(y))^{-1}) = f^2(x)(f(f(y)))^{-1} = f^2(x)(f^2(y))^{-1}$

$= \cancel{(f(x) f(y)}$ $= f(x)(f(y))^{-1} = f(x) f(y^{-1}) = f(xy^{-1})$

So, $xy^{-1} \in K$. So, K is a subgroup of G

Let $g \in G$ and $x \in K$. ~~consider~~ Now,

$f^2(gxg^{-1}) = f(f(gxg^{-1})) = f(f(I_g(x)))$

$= f((f \circ I_g)(x)) = f((I_g \circ f)(x))$ (As $f \circ I_g = I_g \circ f$)

$= f(I_g(f(x))) = f(g f(x) g^{-1}) = f(g) f^2(x) f(g^{-1})$

$= f(g) f(x) f(g^{-1})$ [as $x \in K \Rightarrow f(x) = f^2(x)$]

$= f(gxg^{-1})$. So, $gxg^{-1} \in K$ for all $x \in K$, $g \in G$

So, K is a normal subgroup of G.

(ii) Now G/K is abelian $\Leftrightarrow KxKy = KyKx$ for all $x, y \in G$

$\Leftrightarrow Kxy = Kyx$ for all $x, y \in G$