$\Leftrightarrow\ xyx^{-1}y^{-1} \in K$ for all $x, y \in G$.

Now, $f^2(xyx^{-1}y^{-1}) = f(f(xyx^{-1}y^{-1})) = f(f(I_x(y)\, y^{-1}))$

$= f(f(I_x(y))\, f(y^{-1})) = f((f\circ I_x)(y)\, f(y^{-1})) = f((I_x \circ f)(y)\, f(y^{-1}))$

$= f(I_x(f(y))\, f(y^{-1})) = f(x\, f(y)\, x^{-1}\, f(y^{-1})) = \cancel{f(x) f(f(y)) f(x^{-1}) f(\ldots)}$

$= f(x\, f(y)\, x^{-1} (f(y))^{-1}) = f(x\, I_{f(y)}(x^{-1}))$

$= f(x)\, f(I_{f(y)}(x^{-1})) = f(x)\, I_{f(y)}(f(x^{-1}))$

$= f(x) f(y) f(x^{-1}) (f(y))^{-1} = f(x) f(y) f(x^{-1}) f(y^{-1})$

$= f(xyx^{-1}y^{-1})$

So, $xyx^{-1}y^{-1} \in K$. So, G/K is abelian.

2.1. The notion of direct product is used to factor a group into a product of smaller groups. This factorization gives structural properties of a group. In some cases, it allows for the complete characterization of a certain type of group.

Let $I_n = \{1, 2, \ldots, n\}$, n is a fixed positive integer.

Let $\{G_i : i \in I_n\}$ be a family of groups. Let

$$G = G_1 \times G_2 \times \cdots \times G_n = \{(a_1, a_2, \ldots, a_n) : a_i \in G_i, i \in I_n\}$$

For $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in G$, we defined the binary operation product by

$$(a_1, a_2, \ldots a_n)\cdot(b_1, b_2, \ldots, b_n) = (a_1 b_1, a_2 b_2, \ldots, a_n b_n)$$

In the following theorem, we prove that $(G, \cdot)$ forms a group. We also obtain several important properties of $(G, \cdot)$

Theorem 2.1.1 Let $\{G_i : i \in I_n\}$ be a family of groups and $G = G_1 \times G_2 \times \cdots \times G_n$. Let e_i be the identity of G_i for all $i \in I_n$. Then $(G, \cdot)$ where '$\cdot$' is defined earlier, is a group with $e = (e_1, e_2, \ldots, e_n)$ as the identity element and for all $(a_1, a_2, \ldots, a_n) \in G$, $(a_1, a_2, \ldots, a_n)^{-1} = (a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1})$. Furthermore, let $H_i = \{(e_1, e_2, \ldots, e_{i-1}, a_i, e_{i+1}, \ldots, e_n) : a_i \in G_i\}$ for all $i \in I_n$. Then the following assertions hold:

(i) H_i is a normal subgroup of G for all $i \in I_n$

(ii) For all $a \in G$, a can be uniquely expressed as $a = h_1 h_2 \cdots h_n$ where $h_i \in H_i$, $i \in I_n$

(iii) $H_i \cap (H_1 H_2 \cdots H_{i-1} H_{i+1} \cdots H_n) = \{e\}$ for all $i \in I_n$

(iv) $G = H_1 H_2 \cdots H_n$

Proof: If $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in G$ then

$(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n) = (a_1 b_1, a_2 b_2, \ldots, a_n b_n) \in G$ as

$a_i b_i \in G_i$ for all $i \in I_n$. So this product is a binary operation on G. Now, let $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)$ and $(c_1, c_2, \ldots, c_n) \in G$. Then

$$((a_1, a_2, \ldots a_n)(b_1, b_2, \ldots, b_n))(c_1, c_2, \ldots, c_n) = (a_1 b_1, a_2 b_2, \ldots, a_n b_n)(c_1, c_2, \ldots, c_n)$$

$= ((a_1 b_1) c_1, (a_2 b_2) c_2, \ldots, (a_n b_n) c_n) = (a_1(b_1 c_1), a_2(b_2 c_2), \ldots, a_n(b_n c_n))$ [As G_i are associative for all $i \in I_n$]

$= (a_1, a_2, \ldots, a_n)(b_1 c_1, b_2 c_2, \ldots, b_n c_n)$

$= (a_1, a_2, \ldots, a_n)((b_1, b_2, \ldots, b_n)(c_1, c_2, \ldots, c_n))$. So, the product is associative.

Now $e=(e_1,e_2,\ldots,e_n)\in G$ and for all $a=(a_1,a_2,\ldots,a_n)\in G$,

$$ae=(a_1,a_2,\ldots,a_n)(e_1,e_2,\ldots,e_n)=(a_1e_1,a_2e_2,\ldots,a_ne_n)=(a_1,a_2,\ldots,a_n)=a$$

Similarly $ea=a$. Hence e is the identity in G. To show that every element of G has an inverse in G, let $(a_1,a_2,\ldots,a_n)\in G$. Then $(a_1^{-1},a_2^{-1},\ldots,a_n^{-1})\in G$ since $a_i^{-1}\in G_i$ for all $i\in I_n$ and

$$(a_1,a_2,\ldots,a_n)(a_1^{-1},a_2^{-1},\ldots,a_n^{-1})=(a_1a_1^{-1},a_2a_2^{-1},\ldots,a_na_n^{-1})=(e_1,e_2,\ldots,e_n)=e$$

Similarly, $(a_1^{-1},a_2^{-1},\ldots,a_n^{-1})(a_1,a_2,\ldots,a_n)=e$. So, every element of G has an inverse. Consequently, $(G,\cdot)$ is a group. We also note that by the uniqueness of the inverse element

$$(a_1,a_2,\ldots,a_n)^{-1}=(a_1^{-1},a_2^{-1},\ldots,a_n^{-1}).$$

(i) Let $i\in I_n$. Since $(e_1,e_2,\ldots,e_n)\in H_i$, $H_i\neq\phi$. Let $a=(e_1,\ldots,a_i,\ldots,e_n)$, $b=(e_1,\ldots b_i,\ldots,e_n)\in H_i$. Then

$$ab^{-1}=(e_1,\ldots,a_i,\ldots,e_n)(e_1,\ldots,b_i,\ldots,e_n)^{-1}$$
$$=(e_1,\ldots,a_i,\ldots,e_n)(e_1,\ldots,b_i^{-1},\ldots e_n)=(e_1,\ldots,a_ib_i^{-1},\ldots,e_n)\in H_i$$

as $a_ib_i^{-1}\in G_i$. So, H_i is a subgroup of G. Let $g=(g_1,g_2,\ldots,g_n)\in G$.

Then $$gag^{-1}=(g_1,g_2,\ldots,g_n)(e_1,\ldots a_i,\ldots,e_n)(g_1,g_2,\ldots,g_n)^{-1}$$
$$=(g_1,g_2,\ldots,g_ia_i,\ldots,g_n)(g_1^{-1},g_2^{-1},\ldots,g_n^{-1})$$
$$=(e_1,e_2,\ldots,g_ia_ig_i^{-1},\ldots,e_n)\in H_i \quad\text{as } g_ia_ig_i^{-1}\in G_i.$$

Hence, H_i is a normal subgroup of G, for all $i\in I_n$

(ii) Let $a=(a_1,a_2,\ldots,a_n)\in G$. Let $h_i=(e_1,\ldots,a_i,\ldots,e_n)$ for all $i\in I_n$. Then $a=h_1h_2\cdots h_n$. To show that the represent of a is unique, let $a = k_1k_2 \cdots k_n$ be another representation of a, where $k_i \in H_i$ for all $i \in I_n$. Let $k_i = (e_1, \ldots b_i, \ldots, e_n) \in H_i$ for all $i \in I_n$. Then

$(a_1, a_2, \ldots, a_n) = h_1h_2 \cdots h_n = a = k_1k_2 \cdots k_n = (b_1, b_2, \ldots, b_n)$. This implies that

$a_i = b_i$ for all $i \in I_n$ and so $h_i = k_i$ for all $i \in I_n$. Hence, the representation of a is unique.

(iii) Suppose $a \in H_i \cap (H_1H_2 \cdots H_{i-1}H_{i+1} \cdots H_n)$. Then $a \in H_i$ and $a \in H_1 \cdots H_{i-1}H_{i+1} \cdots H_n$. ~~So, $a = h_1h_2 \cdots h_{i-1}h_{i+1} \cdots h_n$ where~~

Since $a \in H_i$, $a = (e_1, \ldots, a_i, \ldots, e_n) \in H_i$ for some $a_i \in G_i$ and since $a \in H_1 \cdots H_{i-1}H_{i+1} \cdots H_n$, we have $a = h_1h_2 \cdots h_{i-1}h_{i+1} \cdots h_n$ where $h_j = (e_1, \ldots, a_j, \ldots, e_n) \in H_j$ for $j = 1, 2, \ldots, i-1, i+1, \ldots, n$ for some $a_j \in G_j$, $j = 1, 2, \ldots, i-1, i+1, \ldots, n$. So,

$$(e_1, \ldots, a_i, \ldots, e_n) = a = h_1h_2 \cdots h_{i-1}h_{i+1} \cdots h_n = (a_1, \ldots, a_{i-1}, e_i, a_{i+1}, \ldots, a_n)$$

This implies that $a_i = e_i$ for all $i \in I_n$. Hence

$$H_i \cap (H_1H_2 \cdots H_{i-1}H_{i+1} \cdots H_n) = \{e\}$$

(iv) The desired result follows from (ii).

Definition 2.1.2 The group G of Theorem 2.1.1 is called the <u>external direct product</u> of the groups G_i, $i = 1, 2, \ldots, n$

Theorem 2.1.1 motivates the following definition:

Definition 2.1.3 Let G be a group and $\{N_i : i \in I_n\}$ be a family of normal subgroups of G. Then G is called the <u>internal direct product</u> of $N_1, N_2, \ldots, N_n$ if every $a \in G$ can be uniquely expressed as $a = a_1a_2 \cdots a_n$, where $a_i \in N_i$ for all $i \in I_n$.

Let $G = G_1 \times G_2 \times \cdots \times G_n$ be the external direct product of the groups G_i, $i \in I_n$. Let H_i be defined as in Theorem 2.1.1. Then G is the internal direct product of $H_1, H_2, \ldots, H_n$.