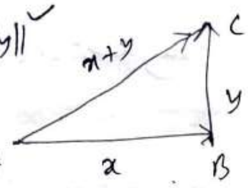
2. Let V be an inner product space and suppose that x and y are orthogonal vectors in V. Prove that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. Deduce the Pythagorian Theorem in $\mathbb{R}^2$.

Solution: $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$

As x, y are orthogonal, $\langle x,y\rangle = 0$, so, $\langle y,x\rangle = 0$

So, $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

In $\mathbb{R}^2$, $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ when x, y are orthogonal gives



$AC^2 = AB^2 + BC^2$ which is Pythagorian Theorem.

as $AB^2 = \|x\|^2$, $BC^2 = \|y\|^2$ and $AC^2 = \|x+y\|^2$.

3. Let $\{v_1, v_2, \dots v_k\}$ be an orthogonal set in an inner product space V and let $a_1, a_2, \dots a_k$ be scalars. Prove that

$$\left\| \sum_{i=1}^{k} a_i v_i \right\|^2 = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2$$

Solution: $\left\| \sum_{i=1}^{k} a_i v_i \right\|^2 = \left\langle \sum_{i=1}^{k} a_i v_i, \sum_{i=1}^{k} a_i v_i \right\rangle$

$$= \sum_{i=1}^{k} \left\langle a_i v_i, \sum_{i=1}^{k} a_i v_i \right\rangle = \sum_{i=1}^{k}\sum_{j=1}^{k} \langle a_i v_i, a_j v_j\rangle$$

$$= \sum_{i=1}^{k} \langle a_i v_i, a_i v_i\rangle + \sum_{\substack{i=1\\ i\neq j}}^{k}\sum_{j=1}^{k} \langle a_i v_i, a_j v_j\rangle$$

$$= \sum_{i=1}^{k} a_i \overline{a_i} \langle v_i, v_i\rangle + \sum_{\substack{i=1\\ i\neq j}}^{k}\sum_{j=1}^{k} a_i \overline{a_j} \langle v_i, v_j\rangle$$

$$= \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2$$

as $\langle v_i, v_j\rangle = 0$ for $i \neq j$ as $\{v_1, v_2, \dots v_n\}$ is an orthogonal set.

4. Let T be a linear operator on an inner product space V, and suppose that $\|T(x)\| = \|x\|$ for all $x \in V$. Prove that T is injective or one-to-one.

Solution: Here $T: V \to V$ is a linear ~~operation~~ operator on V.

Let $T(x) = T(y)$. Then $T(x) - T(y) = \theta \Rightarrow T(x-y) = \theta$ (As T is linear)

or, $\|T(x-y)\| = \|\theta\| = 0$

or, $\|(x-y)\| = 0$ $\left[\text{as } \|T(x)\| = \|x\|\right]$

So, $x - y = \theta$ as $\|x\| = 0$ if and only if $x = \theta$

So, $x = y$. So T is injective.

4.2 The Gram-Schmidt Orthogonalization Process and ~~orth~~ orthogonal Complements

Definition 4.2.1 Let V be an inner product space. A subset of V is an orthonormal basis of V if it is an ordered basis that is orthonormal.

Example: 1. The standard ordered basis of F^n is an orthonormal basis for F^n.

2. The set $\left\{\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)\right\}$ is an orthonormal basis for $\mathbb{R}^2$.

The next theorem and its corollaries illustrate why orthonormal sets and, in particular, orthonormal bases are so important.

Theorem 4.2.2 Let V be an inner product space and $S = \{v_1, v_2, \ldots, v_k\}$ be an ~~orthonormal~~ orthogonal subset of V consisting of non-zero vectors. If $y \in \operatorname{Span}(S) = L(S)$ then

$$y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i$$

Proof: Write $y = \sum_{i=1}^{k} a_i v_i$, $a_1, a_2, \ldots, a_k \in F$. Then for $1 \le j \le k$, we have $\langle y, v_j \rangle = \left\langle \sum_{i=1}^{k} a_i v_i, v_j \right\rangle = \sum_{i=1}^{k} a_i \langle v_i, v_j \rangle$

$= a_j \langle v_j, v_j \rangle$ as $\langle v_i, v_j \rangle = 0$ for $i \neq j$

$= a_j \|v_j\|^2$. So $a_j = \dfrac{\langle y, v_j \rangle}{\|v_j\|^2}$

Corollary 4.2.3 If, in addition to the hypotheses of Theorem 4.2.2, S is orthonormal and $y \in \text{Span}(S)$, then $y = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$

Corollary 4.2.4 Let V be an inner product space and let S be an orthogonal subset of V consisting of non-zero vectors. Then S is linearly independent.

Proof: Suppose that $v_1, v_2, \ldots, v_k \in S$ and $\sum_{i=1}^{k} a_i v_i = \theta$,
As in the proof of Theorem 4.2.2, with $y = \theta$, we have

$$a_j = \frac{\langle \theta, v_j \rangle}{\|v_j\|^2} = 0$$ for all j. So, S is linearly independent.

Example: By Corollary 4.2.4, the orthonormal set

$$\left\{ \frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{3}}(1,-1,1), \frac{1}{\sqrt{6}}(-1,1,2) \right\}$$ obtained

in Example 2 of Page-32, is an orthonormal basis of $\mathbb{R}^3$.
Let $x = (2,1,3)$. The coefficients given by Corollary 4.2.3 that expresses x as linear combination of the basis vectors are

$$a_1 = \frac{1}{\sqrt{2}}(2+1) = \frac{3}{\sqrt{2}} \quad a_2 = \frac{1}{\sqrt{3}}(2-1+3) = \frac{4}{\sqrt{3}}$$ and

$$a_3 = \frac{1}{\sqrt{6}}(-2+1+6) = \frac{5}{\sqrt{6}}$$

As a check, we have

$$(2,1,3) = \frac{3}{2}(1,1,0) + \frac{4}{3}(1,-1,1) + \frac{5}{6}(-1,1,2)$$

Corollary 4.2.4 tells us that the vector space H contains an infinite linearly independent set (Page 32, Example 3) and hence H is not a finite dimensional vector space.

Of course, we have not shown yet that every finite dimensional inner product space possesses an orthonormal basis. The next theorem tells us how to construct an orthogonal set from a linearly independent set of vectors in such a way that both sets generates the same subspace.

Before stating this theorem, let us consider a simple case. Suppose that $\{w_1, w_2\}$ is a linearly independent subset of an inner product space (and hence a basis for some two dimensional subspace. We want to construct ~~from~~ an ~~orthonormal orthogonal set $\{v_1, v_2\}$, $v_1 = w_1$ and $v_2 = w_2 - cw_1$,~~ ~~has this property if~~ an orthogonal set from $\{w_1, w_2\}$ that spans the same subspace. Figure-1 suggests that

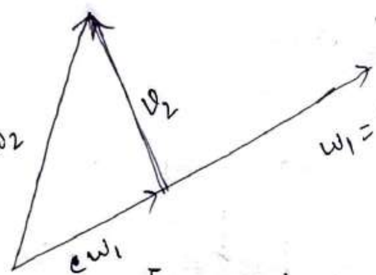


Figure-1

the set $\{v_1, v_2\}$, where $v_1 = w_1$ and $v_2 = w_2 - cw_1$, has this property if c is chosen ~~so that~~ so that v_2 is orthogonal to w_1. To find c, we need only solve the following equation:

$$0 = \langle v_2, w_1\rangle = \langle w_2 - cw_1, w_1\rangle = \langle w_2, w_1\rangle - c\langle w_1, w_1\rangle$$

So, $$c = \frac{\langle w_2, w_1\rangle}{\|w_1\|^2}$$

So, $$v_2 = w_2 - \frac{\langle w_2, w_1\rangle}{\|w_1\|^2} w_1$$

The next theorem shows that this process can be extended to any finite linearly independent subset.

<u>Theorem 4.2.5.</u> Let V be an inner product space and $S = \{w_1, w_2, \ldots, w_n\}$ be a linearly independent subset of V. Define $S' = \{v_1, v_2, \ldots, v_n\}$, where $v_1 = w_1$ and $v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j\rangle}{\|v_j\|^2} v_j$ for $2 \le k \le n$