by Theorem 2.1.1 (ii).

Theorem 2.1.3 Let G be a group and $\{N_i : i \in I_n\}$ be a family of normal subgroups of G. Then G is an internal direct product of $\{N_i : i \in I_n\}$ if and only if $G = N_1 N_2 \cdots N_n$ and $N_i \cap (N_1 N_2 \cdots N_{i-1} N_{i+1} \cdots N_n) = \{e\}$ for all $i \in I_n$, e is the identity element of G.

Proof: Let G be an internal direct product of $\{N_i : i \in I_n\}$. Let $a \in G$. Then $a = a_1 a_2 \cdots a_n$ for some $a_i \in N_i$, $i \in I_n$. So, $a \in N_1 N_2 \cdots N_n$ and this implies that $G = N_1 N_2 \cdots N_n$. We now show that $N_i \cap (N_1 N_2 \cdots N_{i-1} N_{i+1} \cdots N_n) = \{e\}$. Let $i \in I_n$ and $a \in N_i \cap (N_1 \cdots N_{i-1} N_{i+1} \cdots N_n)$. Then $a \in N_i$, $a \in N_1 \cdots N_{i-1} N_{i+1} \cdots N_n$. This implies that we can write $a = a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_n$ for some $a_j \in N_j$, $j \in I_n - \{i\}$. Hence,

$$ee \cdots a \cdots e = a = a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_n$$ are two representations of a, where $a_j \in N_j$, $j \in I_n - \{i\}$. Since the representation of a is unique, $a = e$. Hence $N_i \cap (N_1 N_2 \cdots N_{i-1} N_{i+1} \cdots N_n) = \{e\}$

Conversely, suppose $G = N_1 N_2 \cdots N_n$ and $N_i \cap (N_1 N_2 \cdots N_{i-1} N_{i+1} \cdots N_n) = \{e\}$ for all $i \in I_n$. Then $N_i \cap N_j = \{e\}$ for $i \neq j$. Hence $uv = vu$ for all $u \in N_i$ and $v \in N_j$ (by using the previous result of group theory which says that if H and K are two normal subgroups of a group G and if $H \cap K = \{e\}$ then $hk = kh$ for all $h \in H$ and $k \in K$)

Let $a = a_1 a_2 \cdots a_n = b_1 b_2 \cdots b_n$ be two representations of a, where $a_i, b_i \in N_i$, $i \in I_n$. Then

$$e = a^{-1} a = (a_1 a_2 \cdots a_n)^{-1} (b_1 b_2 \cdots b_n) = a_n^{-1} a_{n-1}^{-1} \cdots a_1^{-1} b_1 b_2 \cdots b_n$$

$= a_1^{-1}b_1 a_2^{-1}b_2 \cdots a_n^{-1}b_n$ since for all $i \neq j$, if $u \in N_i$ and $v \in N_j$ then $uv = vu$. This implies that

$b_i^{-1}a_i = a_1^{-1}b_1 a_2^{-1}b_2 \cdots a_{i-1}^{-1}b_{i-1} a_{i+1}^{-1}b_{i+1} \cdots a_n^{-1}b_n \in N_i \cap (N_1N_2 \cdots N_{i-1}N_{i+1} \cdots N_n)$ for all $i \in I_n$. Since $N_i \cap (N_1N_2 \cdots N_{i-1}N_{i+1} \cdots N_n) = \{e\}$, we must have $b_i^{-1}a_i = e$ or $a_i = b_i$ for all $i \in I_n$. So, a can be written uniquely as $a_1a_2 \cdots a_n$ where $a_i \in N_i$, $i \in I_n$. Hence G is an internal product of $\{N_i : i \in I_n\}$.

In the following theorem, we show that if a group G is an internal direct product of a family of normal subgroups $\{N_i : i \in I_n\}$, then G can be viewed as an external direct product of the groups N_i's.

<u>Theorem 2.1.4</u> Let G be an internal direct product of a family of normal subgroups $\{N_i : i \in I_n\}$. Then

$$G \simeq N_1 \times N_2 \times \cdots \times N_n$$

Proof: Let $a \in G$. Then a can be expressed uniquely as $a_1a_2 \cdots a_n$ where $a_i \in N_i$, $i \in I_n$. Define

$$f : G \longrightarrow N_1 \times N_2 \times \cdots \times N_n \text{ by}$$

$f(a) = (a_1, a_2, \ldots, a_n)$ for all $a \in G$. From the definition of f, it follows that f is well defined and surjective and from the uniqueness of representation of a, it follows that f is injective. We now show that f is a homomorphism. Let $a = a_1a_2 \cdots a_n$ and $b = b_1b_2 \cdots b_n$ be two elements of G, where $a_i, b_i \in N_i$, $i \in I_n$. Now $N_i \cap N_j = \{e\}$ for all $i \neq j$ and so $uv = vu$ for all $u \in N_i$, $v \in N_j$. This implies that $ab = a_1 a_2 \cdots a_n b_1 b_2 \cdots b_n = a_1 b_1 a_2 b_2 \cdots a_n b_n$

So, $f(ab) = (a_1 b_1, a_2 b_2, \ldots, a_n b_n) = (a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n)$
$= f(a)f(b)$ and so f is a homomorphism.

Consequently, f is an ~~iso~~ isomorphism. Hence

$$G \cong N_1 \times N_2 \times \cdots \times N_n$$

Considering Theorem 2.1.4, let us agree to write $G = N_1 \times N_2 \times \cdots \times N_n$ when G is an internal direct product of a family of normal subgroups $\{N_i : i \in I_n\}$

Worked out exercises 2.1.5 : 1. Let G and G_1 be groups and $f: G \to G_1$ be a homomorphism. Let H be a normal subgroup. Suppose that $f/_H : H \to G_1$ is an isomorphism of H onto G_1. Prove that

$$G = H \times \text{Ker} f$$

Proof: Let $a \in G$ Then $f(a) \in G_1 = f(H)$. So, there exists $h \in H$ such that $f(a) = f(h)$. Now $f(a) = f(h)$ implies that $f(h^{-1}a) = e_1$, e_1 is the identity in G_1 and hence $h^{-1}a \in \text{Ker} f$. So, there exists $b \in \text{Ker} f$ such that $b = h^{-1}a$ or $a = hb$. Hence $H = H\,\text{Ker} f$. Suppose $a \in H \cap \text{Ker} f$. Then $a \in H$ and $f(a) = e_1 = f(e)$, e is the identity in G. Since $f/_H$ is injective $f(a) = f(e) \Rightarrow a = e$. So, $H \cap \text{Ker} f = \{e\}$

consequently, $G = H \times \text{Ker} f$

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2. Let G be a group and H and K be subgroups of G such that $G = H \times K$. Let N be a normal subgroup of G such that $N \cap H = \{e\}$ and $N \cap K = \{e\}$, e is the identity in G. Prove that N is commutative.

Solution: Since $G = H \times K$, H and K are normal subgroup of G. ~~such that~~ Now for all $n \in N$, $h \in H$ and $k \in K$, $nh = hn$, $nk = kn$ (by some previous problem on group theory). Let $a, b \in N$ Then there exists $h \in H$ and $k \in K$ such that $b = hk$. Now $ab = a(hk) = (ah)k = (ha)k = h(ak) = h(ka)$ $= (hk)a = ba$. Hence N is commutative.

3. Let G be a group and A and B are subgroups of G. If (i) $G = AB$ (ii) $ab = ba$ for all $a \in A$, $b \in B$ and (iii) $A \cap B = \{e\}$,
prove that G is an internal direct product of A and B

Solution: Let us first show that A and B are normal subgroups of G. For this, let $a \in A$, $g \in G$. There exists $c \in A$ and $b \in B$ such that $g = cb$ by (i).

Now $gag^{-1} = (cb)a(cb)^{-1} = cbab^{-1}c^{-1} = cabb^{-1}c^{-1} = cac^{-1} \in A$

Hence A is a normal subgroup of G. Similarly, B is a normal subgroup of G. Let $g \in G$. Then $g = ab$ for some $a \in A$, $b \in B$. Suppose $g = a_1 b_1$, where $a_1 \in A$, $b_1 \in B$. Then $ab = a_1 b_1$ which implies $a_1^{-1} a = b_1 b^{-1} \in A \cap B$ $= \{e\}$. So, $a = a_1$ and $b = b_1$. So, we find that every element $g \in G$ can be expressed uniquely as $g = ab$, $a \in A$, $b \in B$. Consequently, G is an internal direct product of A, B.

4. Let G be a cyclic group of order mn, where m, n are positive integers such that $\gcd(m, n) = 1$ Show that $G \simeq \mathbb{Z}_m \times \mathbb{Z}_n$