This implies m_2 divides $o(G/K)$. By induction hypothesis, G/K has a subgroup, say H/K of order m_2 where H is a subgroup of G such that $K \subseteq H$. Now,

$$o(H) = o(H/K) \cdot o(K) = m_2 p = m$$

This complete the induction. Hence the theorem is proved.

We now state the fundamental theorem ~~of~~ on finite abelian group without proof.

Theorem 3.1.4 (Fundamental theorem on finite abelian group):
A finite abelian group is direct of cyclic groups of prime ~~pos~~ power order.

Unit-2 Linear Algebra:

4.1 Inner Product Spaces: In the discussion of Inner Product Space, we assume that all vector spaces are over the field F, where denotes either $\mathbb{R}$ or $\mathbb{C}$ (i.e., the field of Real numbers or the field of complex numbers) unless otherwise mentioned.

Definition 4.1.1 Let V be a vector space over F. An inner product on V is a function that assigns, to every ordered pair of vectors x and y in V, a scalar in F, denoted by $\langle x, y\rangle$, such that for all x, y and z in V and all c in F, the following hold:

(a) $\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$

(b) $\langle cx, y\rangle = c\langle x, y\rangle$

(c) $\overline{\langle x, y\rangle} = \langle y, x\rangle$ where bar denotes the complex conjugation.

(d) $\langle x, x\rangle > 0$ if $x \neq 0$

Note that (c) reduces to $\langle x, y\rangle = \langle y, x\rangle$ if $F = \mathbb{R}$. Conditions (a) and (b) simply require that the inner product be linear in the first component. It can be easily shown that

$\left\langle \sum_{i=1}^{n} a_i v_i , y \right\rangle = \sum_{i=1}^{n} a_i \langle v_i , y\rangle$ if $a_1, a_2, \dots, a_n \in F$ and $y, v_1, v_2, \dots, v_n \in V$

Examples 4.1.2 : 1. For $x = (a_1, a_2, \dots, a_n)$ and $y = (b_1, b_2, \dots, b_n)$ in F^n, define

$$\langle x, y\rangle = \sum_{i=1}^{n} a_i \overline{b_i}$$

The verification that $\langle \cdot , \cdot \rangle$ satisfies conditions (a) through (d) is easy. Check it yourself (b), (c) and (d). For (a), if $z = (c_1, c_2, \dots, c_n)$

$$\langle x+z, y\rangle = \sum_{i=1}^{n} (a_i + c_i)\overline{b_i} = \sum_{i=1}^{n} a_i \overline{b_i} + \sum_{i=1}^{n} c_i \overline{b_i} = \langle x, y\rangle + \langle z, y\rangle .$$

Then, for $x = (1+i, 4)$ and $y = (2-3i, 4+5i)$ in $\mathbb{C}^2$

$$\langle x, y\rangle = (1+i)(2+3i) + 4(4-5i) = 15 - 15i$$

This inner product is called the standard inner product on F^n. When $F = \mathbb{R}$, the conjugates are not needed, and in early courses this standard inner product is usually called the dot product or scalar product and is denoted by $x \cdot y$ instead of $\langle x, y\rangle$.

2. If $\langle x, y\rangle$ is any inner product on a vector space V and $r > 0$, we may define another inner product $\langle x, y\rangle' = r\langle x, y\rangle$. If $r \leq 0$ (d) would not hold.

3. Let $V = C([0,1])$, the vector space of real valued continuous functions on $[0,1]$. For $f, g \in V$, define $\langle f, g\rangle = \int_0^1 f(t) g(t)\, dt$

Then $\langle f, g\rangle$ satisfies all the conditions of inner product (Check it.)

Now, we know the definition of conjugate transpose of a matrix $A = [a_{ij}]_{m\times n}$ is a matrix $A^* = [b_{ij}]_{n\times m}$ where $b_{ij} = \overline{a_{ji}}$

If $A = \begin{bmatrix} i & 1+3i \\ 2 & 3+4i \end{bmatrix}$ then $A^* = \begin{bmatrix} -i & 2 \\ 1-3i & 3-4i \end{bmatrix}$

Notice that if x and y are viewed as column vectors in F^n, then $\langle x, y\rangle = y^* x$.

~~Example~~ 4. Let $V = M_{n\times n}(F)$ (the vector space of all $n\times n$ matrices over F)
Define $\langle A, B\rangle = tr(B^*A)$ for $A, B \in V$ (Recall that trace of a matrix $A = [a_{ij}]_{n\times n}$, denoted by $tr(A)$, is defined by $tr(A) = \sum_{i=1}^{n} a_{ii}$.
We verify (a) and (d) of the definition of inner product leave (b) and (c) to you to verify. For this purpose, let $A, B, C \in V$
Then $\langle A+B, C\rangle = tr\left(C^*(A+B)\right) = tr\left(C^*A + C^*B\right) = tr(C^*A) + tr(C^*B)$
$= \langle A, C\rangle + \langle B, C\rangle$
as $tr(A+B) = tr(A) + tr(B)$

Also, let $A = (a_{ij})_{n\times n} \in V$. Then

$\langle A, A\rangle = tr(A^*A) = \sum_{i=1}^{n} b_{ii}$ where ~~the~~ $A^*A = [b_{ij}]_{n\times n}$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n} c_{ik}\, a_{ki}, \quad \text{where } A^* = [c_{ij}]_{n\times n}$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n} \overline{a}_{ki}\, a_{ki} \quad \text{as } c_{ik} = \overline{a_{ki}}$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n} |a_{ki}|^2$$

Now if $A \neq O$, then $a_{ki} \neq 0$ for some k and i. So, $\langle A, A\rangle > 0$

This inner product is called the ~~Frobenius~~ Frobenius inner product.

A vector space V over F endowed with a specific inner product is called an inner product space. If $F = \mathbb{C}$, we call V a complex inner product space, whereas if $F = \mathbb{R}$, we call V a real inner product space.
It is clear that if V has an inner product $\langle x, y\rangle$ and W is a subspace of V, then W is also an inner product space when the same function $\langle x, y\rangle$ is restricted to the vectors $x, y \in W$
So, Examples 1, 3 and 4 also provide examples of inner product spaces.

From now onward, F^n will denote the inner product space with the standard inner product as defined in example 1. Likewise, $M_{n\times n}(F)$ will denote the inner product space with the Frobenius inner product as defined in example 4. Students are cautioned that two distinct inner products on a given vector space yeild two distinct inner product spaces. For instance, it can be shown that both $\langle f(x), g(x)\rangle_1 = \int_0^1 f(t)g(t)\,dt$ and $\langle f(x), g(x)\rangle_2 = \int_{-1}^1 f(t)g(t)\,dt$ are inner product on the vector space $P(\mathbb{R})$ of polynomials with real coefficients. Even though the underlying vector space is the same, however, these two inner products yeilds two inner product spaces. For example, for the polynomials $f(x) = x$ and $g(x) = x^3$

$\langle f(x), g(x)\rangle_2 = 0$ but $\langle f(x), g(x)\rangle_1 \neq 0$

A very important inner product space that resembles $C([0,1])$ is the space H of continuous complex valued functions defined on the interval $[0, 2\pi]$ with the inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,dt$$

This inner product space ~~which~~ arises often in the context of physical situations. ~~in examples~~

At this point, we mention a few facts about ~~complex integration~~ integration of complex valued functions. First, the imaginary number i can be treated as constant under the integral sign. Second, every complex valued function f may be written as $f = f_1 + i f_2$ where f_1 and f_2 are real valued functions. Thus we have

$$\int f = \int f_1 + i\int f_2 \quad \text{and} \quad \overline{\int f} = \int \overline{f}$$

From these properties, as well as the assumption of continuity, it follows that H is an inner product space.

Some properties that follow easily from the definition of inner product are contained in the next theorem.