Theorem 4.1.3 Let V be an inner product space. Then for $x, y, z \in V$ and $c \in F$, the following statements are true:

(a) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

(b) $\langle x, cy \rangle = \bar{c}\langle x, y \rangle$

(c) $\langle x, \theta \rangle = \langle \theta, x \rangle = 0$, θ is the null vector

(d) $\langle x, x \rangle = 0$ if and only if $x = \theta$

(e) If $\langle x, y \rangle = \langle x, z \rangle$ for all $x \in V$, then $y = z$

Proof: (a) $\langle x, y+z \rangle = \overline{\langle y+z, x \rangle} = \overline{\langle y, x \rangle + \langle z, x \rangle}$

$= \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle} = \langle x, y \rangle + \langle x, z \rangle$

(b) $\langle x, cy \rangle = \overline{\langle cy, x \rangle} = \overline{c\langle y, x \rangle} = \bar{c}\,\overline{\langle y, x \rangle} = \bar{c}\langle x, y \rangle$

The proofs of (c), (d) and (e) are left as exercises.

In order to generalize the notion of length in $\mathbb{R}^3$ to arbitrary inner product spaces, we need only to observe that the length of $x = (a, b, c) \in \mathbb{R}^3$ is given by $\sqrt{a^2+b^2+c^2} = \sqrt{\langle x, x \rangle}$. This leads to the following definition.

Definition 4.1.4. Let V be an inner product space. For $x \in V$, we define the norm or length of x by $\|x\| = \sqrt{\langle x, x \rangle}$

Example: Let $V = F^n$. If $x = (a_1, a_2, \cdots, a_n)$ then

$$\|x\| = \|(a_1, a_2, \cdots, a_n)\| = \left[\sum_{i=1}^{n} |a_i|^2\right]^{1/2}$$ is the Euclidean

definition of length. Note that if $n=1$, we have $\|a\| = |a|$

Theorem 4.1.5 Let V be an inner product space over F. Then for all $x, y \in V$ and $c \in F$, the following statements are true:

(a) $\|cx\| = |c|\,\|x\|$.

(b) $\|x\| = 0$ if and only if $x = \theta$. In any case $\|x\| \geq 0$.

(c) (Cauchy Schwarz Inequality) $|\langle x, y \rangle| \leq \|x\|\,\|y\|$.

(d) (Triangle Inequality) $\|x+y\| \leq \|x\| + \|y\|$.

Proof: We leave the proofs of (a) and (b) as exercises.

(c) If $y=\theta$, then $\|y\|=0$ and $\langle x,\theta\rangle=0$. So, $\langle x,y\rangle=\|x\|\|y\|$. So, assume, $y\neq\theta$. For any $c\in F$, we have

$$0\le\|x-cy\|^2=\langle x-cy,x-cy\rangle=\langle x,x-cy\rangle-c\langle y,x-cy\rangle$$
$$=\langle x,x\rangle-\bar{c}\langle x,y\rangle-c\langle y,x\rangle+c\bar{c}\langle y,y\rangle$$

In particular, if we set $c=\dfrac{\langle x,y\rangle}{\langle y,y\rangle}$,

the inequality becomes,

$$0\le\langle x,x\rangle-\frac{\overline{\langle x,y\rangle}\langle x,y\rangle}{\langle y,y\rangle}-\frac{\langle x,y\rangle}{\langle y,y\rangle}\langle y,x\rangle+\frac{\langle x,y\rangle}{\langle y,y\rangle}\frac{\overline{\langle x,y\rangle}}{\langle y,y\rangle}\langle y,y\rangle$$

or, $0\le\langle x,x\rangle-\dfrac{|\langle x,y\rangle|^2}{\langle y,y\rangle}-\dfrac{\langle x,y\rangle\overline{\langle x,y\rangle}}{\langle y,y\rangle}+\dfrac{|\langle x,y\rangle|^2}{\langle y,y\rangle}$ [as $\langle y,y\rangle\neq0$]

or, $0\le\langle x,x\rangle-\dfrac{|\langle x,y\rangle|^2}{\langle y,y\rangle}-\dfrac{|\langle x,y\rangle|^2}{\langle y,y\rangle}+\dfrac{|\langle x,y\rangle|^2}{\langle y,y\rangle}$

or, $0\le\langle x,x\rangle-\dfrac{|\langle x,y\rangle|^2}{\langle y,y\rangle}$

or, $|\langle x,y\rangle|^2\le\|x\|^2\|y\|^2$

or, $|\langle x,y\rangle|\le\|x\|\|y\|$

(d) We have, $\|x+y\|^2=\langle x+y,x+y\rangle=\langle x,x\rangle+\langle y,x\rangle+\langle x,y\rangle+\langle y,y\rangle$

$=\langle x,x\rangle+\langle x,y\rangle+\overline{\langle x,y\rangle}+\langle y,y\rangle$

$=\langle x,x\rangle+2\,\mathrm{Re}\langle x,y\rangle+\langle y,y\rangle$ [$\mathrm{Re}\langle x,y\rangle$ is the real part of $\langle x,y\rangle$]

$\le\langle x,x\rangle+2|\langle x,y\rangle|+\langle y,y\rangle$

$\le\langle x,x\rangle+2\|x\|\|y\|+\langle y,y\rangle$ [By Cauchy Schwarz Inequality]

$=\|x\|^2+2\|x\|\|y\|+\|y\|^2$

$=(\|x\|+\|y\|)^2$

or, $\|x+y\|^2\le\|x\|+\|y\|$

Example: For F^n, we may apply (c) and (d) of Theorem 4.1.5 to the standard inner product to obtain the following known inequalities:

$$\left|\sum_{i=1}^{n} a_i \bar{b}_i\right| \leq \left(\sum_{i=1}^{n}|a_i|^2\right)^{1/2}\left(\sum_{i=1}^{n}|b_i|^2\right)^{1/2}$$

and $$\left(\sum_{i=1}^{n}|a_i+b_i|^2\right)^{1/2} \leq \left(\sum_{i=1}^{n}|a_i|^2\right)^{1/2} + \left(\sum_{i=1}^{n}|b_i|^2\right)^{1/2}$$

The student may remember that from earlier courses that, for x and y in $\mathbb{R}^3$ or $\mathbb{R}^2$, we have that $\langle x,y\rangle = \|x\|\|y\|\cos\theta$ where θ $(0 \leq \theta \leq \pi)$ denotes the angle between x and y. This equation implies (c) immediately since $|\cos\theta| \leq 1$. Notice also that non-zero ~~a~~ vectors x and y are perpendicular if and only if $\cos\theta = 0$, that is, if and only if $\langle x,y\rangle = 0$.

We are now at the point where we can generalize the notion of perpendicularity to arbitrary inner product spaces.

Definitions 4.1.6 Let V be an inner product space. Vectors x and y in V are ~~b~~ orthogonal (perpendicular) if $\langle x,y\rangle = 0$. A subset S of V is orthogonal if any two distinct vectors in S are orthogonal. A vector x in V is a unit vector if $\|x\| = 1$. Finally a subset S of V is orthonormal if S is orthogonal and consists entirely of unit vectors.

Note that if $S = \{v_1, v_2, \ldots\}$, then S is ~~orthonor~~ orthonormal if $\langle v_i, v_j\rangle = \delta_{ij}$ where δ_{ij} denotes the Kronecker delta, i.e.,

$$\delta_{ij} = 0, \ i \neq j$$
$$= 1, \ i = j$$

Also observe that multiplying vectors by non-zero scalars does not affect their orthogonality and that if x is any non-zero vector, then $\left(\frac{1}{\|x\|}\right)x$ is a unit vector. The process of multiplying a non-zero vector by the reciprocal of its length is called normalizing.

Example 2 In F^3, $\{(1,1,0), (1,-1,1), (-1,1,2)\}$ is an orthogonal set of non-zero vectors, but it is not orthonormal; however, if we normalize the vectors, we obtain the orthonormal set

$$\left\{ \frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{3}}(1,-1,1), \frac{1}{\sqrt{6}}(-1,1,2) \right\}.$$

Our next example is of an infinite orthonormal ~~set~~ set that is important in analysis.

Example 3 Recall the inner product space H (defined on Page-28). We introduce an important orthonormal subset ~~of~~ S of H. For what follows, i is the imaginary number such that $i^2 = -1$. For any integer n, let $f_n(t) = e^{int}$ where $0 \le t \le 2\pi$. [Recall that $e^{int} = \cos nt + i \sin nt$] Now define $S = \{f_n : n \text{ is an integer}\}$.

Clearly S is a subset of H. Using the property that $\overline{e^{it}} = e^{-it}$ for every real number t, we have for $m \neq n$,

$$\langle f_m, f_n \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{imt}\, \overline{e^{int}}\, dt = \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)t}\, dt$$

$$= \frac{1}{2\pi(m-n)} \left[e^{i(m-n)t} \right]_0^{2\pi} = 0.$$

Also $\langle f_n, f_n \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-n)t}\, dt = \frac{1}{2\pi} \int_0^{2\pi} dt = 1$

In other words, $\langle f_m, f_n \rangle = \delta_{ij}$

Worked out exercises 4.1.7 : 1. Prove the parallelogram of an inner product space V, i.e., show that

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \text{ for all } x, y \in V$$

Proof: $\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y \rangle + \langle x-y, x-y \rangle$

$= \langle x,x \rangle + \langle x,y \rangle + \langle y,x \rangle + \langle y,y \rangle + \langle x,x \rangle - \langle x,y \rangle - \langle y,x \rangle + \langle y,y \rangle$

$= 2\langle x,x \rangle + 2\langle y,y \rangle = 2\|x\|^2 + 2\|y\|^2$

Note: In $\mathbb{R}^2$, this gives $AC^2 + BD^2 = 2(AB^2 + AD^2)$ for the parallelogram $\square ABCD$