Then S' is an orthogonal set of non-zero vectors such that $span(S') = span(S)$.

Proof: The proof is by mathematical induction n, the number of vectors in S. For $k=1,2,\ldots,n$, let $S_k = \{w_1, w_2, \ldots, w_k\}$. In $n=1$, then the theorem is proved by taking $S_1' = S_1$, i.e., $v_1 = w_1 \neq 0$. Assume that then that $S_{k-1}' = \{v_1, v_2, \ldots, v_{k-1}\}$, with the desired properties has been constructed by the repeated use of (1). We show that $S_k' = \{v_1, v_2, \ldots, v_{k-1}, v_k\}$ also has the ~~to~~ desired properties, where v_k is obtained from S_{k-1}' by (1). If $v_k = 0$, then (1) implies that $w_k \in span(S_{k-1}') = span(S_{k-1})$, which contradicts the assumption that S_k is linearly independent. For $1 \leq i \leq k-1$, it follows from (1) that

$$\langle v_k, v_i\rangle = \langle w_k, v_i\rangle - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j\rangle}{\|v_j\|^2} \langle v_j, v_i\rangle$$

$$= \langle w_k, v_i\rangle - \frac{\langle w_k, v_i\rangle}{\|v_i\|^2} \|v_i\|^2 = 0$$

Since $\langle v_j, v_i\rangle = 0$ for $j \neq i$ by the induction assumption that S_{k-1}' is orthogonal. Hence S_k' is an orthogonal set of non-zero vectors. Now by (1), we have that $span(S_k') \subseteq span(S_k)$. But by corollary 4.2.4 of Theorem 4.2.2, S_k' is linearly independent; so $\dim(span(S_k')) = \dim(span(S_k)) = k$. So, $span(S_k') = span(S_k)$.

The construction of $\{v_1, v_2, \ldots, v_n\}$ by the use of Theorem 4.2.5 is called the Gram-Schmidt orthogonalization process.

Example: 1. In $\mathbb{R}^4$, let $w_1 = (1,0,1,0)$, $w_2 = (1,1,1,1)$ and $w_3 = (0,1,2,1)$. Then $\{w_1, w_2, w_3\}$ is linearly independent.

we use Gram - Schmidt process to compute the orthogonal vectors v_1, v_2 and v_3 and then we normalize these vectors to obtain an orthonormal set.

Take $v_1 = w_1 = (1,0,1,0)$ Then

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$= (1,1,1,1) - \frac{2}{2}(1,0,1,0) = (0,1,0,1).$$

Finally $v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$

$$= (0,1,2,1) - \frac{2}{2}(1,0,1,0) - \frac{2}{2}(0,1,0,1)$$

$$= (-1,0,1,0)$$

These vectors can be normalized to obtain the orthonormal set $\{u_1, u_2, u_3\}$ where

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}(1,0,1,0)$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}}(0,1,0,1)$$

and $u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{2}}(-1,0,1,0)$

Example 2. Let $V = P(\mathbb{R})$ be the vector space of all polynomials with real coefficients with the inner product $\langle f(x), g(x) \rangle = \int_{-1}^{1} f(t) g(t)\, dt$, and consider the subspace $P_2(\mathbb{R})$ (subspace of polynomials with real coefficients of degree less or equal to 2) with the standard ordered basis $B = \{1, x, x^2\}$. We use the Gram-Schmidt process to replace B by an orthogonal basis $\{v_1, v_2, v_3\}$ for $P_2(\mathbb{R})$, and use this orthogonal basis to obtain an orthonormal basis for $P_2(\mathbb{R})$.

Take $v_1 = 1$ Then $\|v_1\|^2 = \int_{-1}^{1} 1^2 dt = 2$ and $\langle x, v_1\rangle = \int_{-1}^{1} t.1\, dt = 0$

So, $v_2 = x - \frac{\langle x, v_1\rangle}{\|v_1\|^2} = x - \frac{0}{2} = x$

Further $\langle x^2, v_1\rangle = \int_{-1}^{1} t^2.1\, dt = \frac{2}{3}$ and $\langle x^2, v_2\rangle = \int_{-1}^{1} t^2.t\, dt = 0$

So, $v_3 = x^2 - \frac{\langle x^2, v_1\rangle}{\|v_1\|^2} - \frac{\langle x^2, v_2\rangle}{\|v_2\|^2} = x^2 - \frac{2}{3}$

So, we conclude that $\{1, x, x^2 - \frac{2}{3}\}$ is an orthogonal basis for $P_2(\mathbb{R})$

Now $\|1\|^2 = \int_{-1}^{1} 1.dt = 2$, $\|x\|^2 = \int_{-1}^{1} t^2 dt = \frac{2}{3}$

$$\|x^2 - \tfrac{2}{3}\|^2 = \int_{-1}^{1}\left(t^2 - \tfrac{2}{3}\right)^2 dt = \int_{-1}^{1}\left(t^4 - \tfrac{4}{3}t^2 + \tfrac{4}{9}\right) dt$$

$$= \frac{2}{5} - \frac{4}{3}\times\frac{2}{3} + \frac{4}{9}\times 2 = \frac{2}{5} - \frac{8}{9} + \frac{8}{9} = \frac{2}{5}$$

To obtain an orthonormal basis, we normalize v_1, v_2 and v_3 to obtain $u_1 = \frac{1}{\sqrt{2}}$, $u_2 = \sqrt{\frac{3}{2}}\, x$

and $u_3 = \sqrt{\frac{5}{2}}\left(x^2 - \frac{2}{3}\right) = \sqrt{\frac{5}{18}}\left(3x^2 - 2\right)$

Then $\{u_1, u_2, u_3\}$ is the desired orthonormal basis for $P_2(\mathbb{R})$.

Theorem 4.2.6. Let V be a non-zero finite dimensional inner product process. Then V has an orthonormal basis B. Furthermore, if $B = \{v_1, v_2, \dots, v_n\}$ and $x \in V$, Then
$$x = \sum_{i=1}^{n} \langle x, v_i\rangle v_i .$$

Proof: Let B_0 be an ordered basis for V. Apply Theorem 4.2.5 to obtain an orthonormal set B' of non-zero vectors with span $(B') = $ span $(B_0) = V$. By normalizing each vector in B', we obtain an orthonormal set B that generates V. By corollary 4.2.4 to Theorem 4.2.2, B is linearly independent; therefore B is an orthonormal basis for V.
• The remainder of the theorem follows from Corollary 4.2.3 of Theorem 4.2.2.

~~Example 2~~

Corollary 4.2.7 Let V be a finite dimensional inner product space with an orthonormal basis $B = \{v_1, v_2, \ldots, v_n\}$. Let T be a linear operator on V, and let $A = [T]_B$, the matrix of T with respect to the ordered basis B.

Then for any i and j, $a_{ij} = \langle T(v_j), v_i \rangle$. where $A = [a_{ij}]_{n\times n}$.

Proof: From Theorem 4.2.6, we have

$$T(v_j) = \sum_{i=1}^{n} \langle T(v_j), v_i \rangle v_i .$$

Hence $a_{ij} = \langle T(v_j), v_i \rangle$

Definition 4.2.8 (Orthogonal Complement) Let S be a non-empty subset of an inner product space V. We define $S^\perp$ (read "S perp") to be the set of all vectors in V that are orthogonal to every vector in S; i.e., $S^\perp = \{x \in V : \langle x, y \rangle = 0 \text{ for all } y \in S\}$. The set $S^\perp$ is called the orthogonal complement of S.