$\theta \in S^\perp$ as $\langle \theta, y\rangle = 0$ for all $y \in S$. Let $\alpha, \beta \in S^\perp$

Then $\langle \alpha, y\rangle = 0$ and $\langle \beta, y\rangle = 0$ for all $y \in S$

So, $\langle \alpha+\beta, y\rangle = \langle \alpha, y\rangle + \langle \beta, y\rangle = 0$ for all $y \in S$

So, $\alpha+\beta \in S^\perp$. Let $x \in S^\perp$ and $c \in F$,

then $\langle x, y\rangle = 0$ for all $y \in S$. So, $\langle cx, y\rangle = c\langle x, y\rangle = c.0 = 0$

for all $y \in S$. So, $cx \in S^\perp$.

So, $S^\perp$ is a subspace of V for any subset S of V.

Exercise: Show that $\{\theta\}^\perp = V$ and $V^\perp = \{\theta\}$ for any inner product space V.

Example: Let $V = \mathbb{R}^3$ and $S = \{(0,0,1)\}$

Then
$$S = \{\alpha \in \mathbb{R}^3 : \langle \alpha, (0,0,1)\rangle = 0\}$$
$$= \{\alpha = (x,y,z) \in \mathbb{R}^3 : \langle (x,y,z), (0,0,1)\rangle = 0\}$$
$$= \{(x,y,z) \in \mathbb{R}^3 : z = 0\} = xy\text{-plane}.$$

Theorem 4.2.9 Let β be a basis for a subspace W of an inner product space V, and let $z \in V$. Prove that $z \in W^\perp$ if and only if $\langle z, v\rangle = 0$ for every $v \in \beta$.

Proof: Let $\langle z, v\rangle = 0$ for every $v \in \beta$. Let $y \in W$

Then $y = \sum_{i=1}^{k} c_i v_i$, $v_i \in \beta$, $i = 1, 2, \ldots, k$

So, $\langle z, y\rangle = \langle z, \sum_{i=1}^{k} c_i v_i\rangle = \sum_{i=1}^{k} \langle z, c_i v_i\rangle = \sum_{i=1}^{k} \bar{c_i}\langle z, v_i\rangle$

$= 0$ as $\langle z, v_i\rangle = 0$ as $v_i \in \beta$, $i = 1, 2, \ldots, k$

So, $z \in W^\perp$. Conversely, let $z \in W^\perp$. Then $\langle z, y\rangle = 0$ for all $y \in W$. As β is a basis for the subspace W

So, each $v \in \beta$. So, $\langle z, v \rangle = 0$ for each $v \in \beta$.

~~Theorem 4.2.10 Suppose that $S = \{v_1, v_2, \dots, v_k\}$ is an~~
~~orthonormal set in an n-dimensional inner product space V.~~
~~Then (a) S can be exte~~

Theorem 4.2.10 Let W be a finite dimensional subspace of an inner product space V, and let $y \in V$. Then there exists unique vectors $u \in W$ and $z \in W^{\perp}$ such that $y = u + z$. Furthermore, if $\{v_1, v_2, \dots, v_k\}$ is an orthonormal basis for W, then

$$u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$$

Proof: Let $\{v_1, v_2, \dots, v_k\}$ be an orthonormal basis for W. Let $u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$ and let $z = y - u$. Clearly $u \in W$ and $y = u + z$. To show that $z \in W^{\perp}$, it suffices to show by Theorem 4.2.9, that z is orthogonal to each v_j, $j = 1, 2, \dots, n$. For any j, we have

$$\langle z, v_j \rangle = \left\langle y - \sum_{i=1}^{k} \langle y, v_i \rangle v_i, v_j \right\rangle = \langle y, v_j \rangle - \sum_{i=1}^{k} \langle y, v_i \rangle \langle v_i, v_j \rangle$$

$$= \langle y, v_j \rangle - \langle y, v_j \rangle \quad [\text{As } \langle v_i, v_j \rangle = 0 \ i \neq j \text{ and } \langle v_j, v_j \rangle = 1]$$

$$= 0$$

~~Thus~~ To show ~~that~~ the uniqueness of u and z. Suppose that $z = u + z = u' + z'$, where $u, u' \in W$ and $z, z' \in W^{\perp}$. Then $u - u' = z' - z \in W \cap W^{\perp} = \{0\}$. So, $u = u'$, $z = z'$.

Corollary 4.2.11 In the notation of theorem 4.2.10, the vector u is the unique vector in W that is "closest" to y; that is, for any $x \in W$ $\|y - x\| \geq \|y - u\|$

and this inequality is an equality if and only if $x = u$

Proof: As in Theorem 4.2.10, we have that $y = u + z$, $z \in W^{\perp}$. Let $x \in W$. Then $u - x$ is orthogonal to z. So, we have

$$\|y-x\|^2 = \|u+z-x\|^2 = \|(u-x)+z\|^2 = \|u-x\|^2 + \|z\|^2$$
$$\geq \|z\|^2 = \|y-u\|^2 \text{. So, } \|y-x\| \geq \|y-u\|$$

Now suppose, $\|y-x\| = \|y-u\|$. So, the inequality above becomes an equality, and therefore $\|u-x\|^2 + \|z\|^2 = \|z\|^2$

or, $\|u-x\|^2 = 0$ or, $\|u-x\| = 0$ or, $x = u$

Conversely, If $x = u$ then $(x-u) = \theta$

So, $\|y-x\|^2 = \|(u-x)+z\|^2 = \|z\|^2 = \|y-u\|^2$

or, $\|y-x\| = \|y-u\|$

The vector u in the corollary 4.2.11 is called the orthogonal projection of y on W.

Theorem 4.2.12 Suppose $S = \{v_1, v_2, \ldots, v_k\}$ is an orthonormal set in an n-dimensional inner product space V. Then (a) S can be extended to an orthonormal basis $\{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V

(b) If $V = \mathrm{Span}(S)$ then $S_1 = \{v_{k+1}, v_{k+2}, \ldots, v_n\}$ is an orthonormal basis for $W^{\perp}$

(c) If W is any subspace of V. Then $\dim(V) = \dim(W) + \dim(W^{\perp})$

Proof: As any linearly independent subset of a finite dimensional vector space can be extended to a basis of that space, so, S can be extended to a basis $S' = \{v_1, v_2, \ldots, v_k, w_{k+1}, \ldots, w_n\}$ for V. Now apply the Gram Schmidt orthogonalization process on S'. Now using mathematical induction, we can prove that if $\{u_1, u_2, \ldots, u_n\}$ be an orthogonal set of non-zero vectors then the vectors $\{x_1, x_2, \ldots, x_n\}$ derived from the Gram Schmidt process satisfy $x_i = u_i$, for $i = 1, 2, \ldots, n$. So, using this result, the first k vectors resulting from this process are the vectors in S_1 and this new set spans V. Normalizing the last $n-k$ vectors of this set produces an orthonormal set that spans V. So, the result now follows.

(b) Because S_1 is a subset of a basis, it is linearly independent. Since S_1 is clearly a subset of $W^\perp$, we need only show that S_1 spans $W^\perp$. Note that for any $x \in V$, we have

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$$

If $x \in W^\perp$, then $\langle x, v_i \rangle = 0$ for $i = 1, 2, \ldots, k$

$$\text{So,} \quad x = \sum_{i=k+1}^{n} \langle x, v_i \rangle v_i \in \text{Span}(S_1)$$

(c) Let W be a subspace of V. It is a finite dimensional inner product vector space, as V is and so it has an orthonormal basis $\{v_1, v_2, \ldots, v_k\}$. By (a) and (b), we have

$$\dim(V) = n = k + (n-k) = \dim(W) + \dim(W^\perp)$$