Example: Let $W = span(\{e_1, e_2\})$ in F^3 where $e_1 = (1,0,0)$ and $e_2 = (0,1,0)$.
Then ~~$x \in$~~ $x = (a,b,c) \in W^\perp$ if and only if $0 = \langle x, e_1\rangle = a$ and $0 = \langle x, e_2\rangle = b$. So, $x = (0,0,c)$ and therefore $W^\perp = span(\{e_3\})$ where $e_3 = (0,0,1)$ and from (c), that $\dim(W^\perp) = 3-2 = 1$

Theorem (Bessel's Inequality): Let V be an inner product space, and let $S = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal subset of V. ~~Prove that~~ Then for any $x \in V$ we have

$$\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i\rangle|^2$$

Proof: Let $W = span(S)$. Let $x \in V$. By the Theorem 4.2.10, $\exists$ unique vectors $y \in W$ and $z \in W^\perp$ such that $x = y + z$ and as $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis of W, $y = \sum_{i=1}^{n} \langle x, v_i\rangle v_i$.

Now $\|x\|^2 = \|y+z\|^2 = \|y\|^2 + \|z\|^2$ as $\langle y, z\rangle = 0$

So, $\|x\|^2 \geq \|y\|^2$

Now $\|y\|^2 = \langle y, y\rangle = \left\langle \sum_{i=1}^{n} \langle x, v_i\rangle v_i, \sum_{i=1}^{n} \langle x, v_i\rangle v_i \right\rangle$

$$= \sum_{i=1}^{n} |\langle x, v_i\rangle|^2 \quad (\text{As } \{v_1, v_2, \ldots, v_n\} \text{ are orthonormal set})$$

So, $\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i\rangle|^2$.

Note: The Bessel's inequality is an equality if and only if $x \in Span(S) = W$

If $x \in Span(S) = W$ Then $x = x + \theta$, $\theta \in W^\perp$

So, $\|x\|^2 = \sum_{i=1}^{n} |\langle x, v_i\rangle|^2$ from the previous result.

Conversely, let $\|x\|^2 = \sum_{i=1}^{n} |\langle x, v_i\rangle|^2$, So, $\|z\| = 0 \Rightarrow z = \theta$

So, $x = \sum_{i=1}^{n} \langle x, v_i\rangle v_i \Rightarrow x \in span(S)$.

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4.3 The Adjoint of a Linear operator.

We know the concept of the conjugate transpose A^* of a matrix A. For a linear operator on an inner product space V, we now define a related linear operator on V called the adjoint of T, whose matrix representation with respect to any orthonormal basis β for V is $[T]_\beta^*$ where $[T]_\beta$ is the matrix of T with respect to the basis β. The analogy between conjugation of complex numbers and adjoints of linear operators will become apparent. We first need a preliminary result.

Let V be an inner product space, and let $y \in V$. The function $g: V \to F$ defined by $g(x) = \langle x, y \rangle$ is clearly linear. More interesting is the fact that if V is finite dimensional, every linear transformation from V into F is of this form.

Theorem 4.3.1 Let V be a finite dimensional inner product space over F, and let $g: V \to F$ be a linear transformation. Then there exists a unique vector $y \in V$ such that $g(x) = \langle x, y \rangle$ for all $x \in V$.

Proof: Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V, and let $y = \sum_{i=1}^{n} \overline{g(v_i)}\, v_i$

Define $h: V \to F$ by $h(x) = \langle x, y \rangle$, which is clearly linear (check it). Furthermore, for $1 \le j \le n$, we have $h(v_j) = \langle v_j, y \rangle = \langle v_j, \sum_{i=1}^{n} \overline{g(v_i)}\, v_i \rangle$

$= \sum_{i=1}^{n} g(v_i)\langle v_j, v_i\rangle = \sum_{i=1}^{n} g(v_i)\delta_{ji} = g(v_j)$.

Since g and h both agree on β, we have $g = h$.

To show that y is unique, suppose that $g(x) = \langle x, y'\rangle$ for all x. Then $\langle x, y\rangle = \langle x, y'\rangle$ for all $x \in V$. So by Theorem 4.1.3 (e) of page-29, we get $y = y'$

Example: Define $g: \mathbb{R}^2 \to \mathbb{R}$ by $g(a_1, a_2) = 2a_1 + a_2$. Clearly g is a linear transformation. Let $\beta = \{(1,0), (0,1)\}$ and let $y = g((1,0))(1,0) + g((0,1))(0,1) = (2,1)$ as in the proof of the theorem 4.3.1. Then $g(a_1, a_2) = \langle (a_1, a_2), (2,1)\rangle = 2a_1 + a_2$

Theorem 4.3.2 Let V be a finite dimensional inner product space, and let T be a linear operator on V. Then there exists a unique function $T^*: V \to V$ such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in V$. Furthermore T is linear.

Proof: Let $y \in V$. Define $g: V \to F$ by $g(x) = \langle T(x), y\rangle$ for all $x \in V$. We first show that g is linear. Let $x_1, x_2 \in V$ and $c \in F$. $g(cx_1 + x_2) = \langle T(cx_1 + x_2), y\rangle$
$= \langle cT(x_1) + T(x_2), y\rangle = c\langle T(x_1), y\rangle + \langle T(x_2), y\rangle = cg(x_1) + g(x_2)$.

Hence g is linear.

We now apply Theorem 4.3.1 to obtain a unique vector $y' \in V$ such that $g(x) = \langle x, y'\rangle$; that is, $\langle T(x), y\rangle = \langle x, y'\rangle$ for all $x \in V$. Defining $T^*: V \to V$ by $T^*(y) = y'$, we have $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$. To show that

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T^* is ~~linear~~ linear, let $y_1, y_2 \in V$ and $c \in F$. Then for any $x \in V$, we have

$$\langle x, T^*(cy_1+y_2)\rangle = \langle T(x), cy_1+y_2\rangle$$
$$= \bar{c}\langle T(x), y_1\rangle + \langle T(x), y_2\rangle$$
$$= \bar{c}\langle x, T^*(y_1)\rangle + \langle x, T^*(y_2)\rangle$$
$$= \langle x, cT^*(y_1)\rangle + \langle x, T^*(y_2)\rangle$$
$$= \langle x, cT^*(y_1) + T^*(y_2)\rangle$$

Since x is arbitrary, $T^*(cy_1+y_2) = cT^*(y_1) + T^*(y_2)$ by Theorem 4.1.3 (e) of page-29.

Finally, we need to show that T^* is unique. Suppose $S : V \to V$ is linear and it satisfies $\langle T(x), y\rangle = \langle x, S(y)\rangle$ for all $x, y \in V$. Then $\langle x, T^*(y)\rangle = \langle x, S(y)\rangle$ for all $x, y \in V$, so $T^* = S$.

— The linear operator T^* described in Theorem 4.3.2 is called the adjoint of the operator T. The symbol T^* is read "T star". Thus T^* is the unique operator on V satisfying $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in V$. Note that we also have

$$\langle x, T(y)\rangle = \overline{\langle T(y), x\rangle} = \overline{\langle y, T^*(x)\rangle} = \langle T^*(x), y\rangle$$

so $\langle x, T(y)\rangle = \langle T^*(x), y\rangle$ for all $x, y \in V$. We may view these equations symbolically as adding a $*$ to T when shifting its position inside the inner product symbol.

For an infinite dimensional inner product space, the adjoint of a linear operator T may be defined to be the function