T^* such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in V$, provided it exists. Although the uniqueness and linearity of T^* follows as before, the existence of the adjoint is not guaranteed. The reader should observe the necessity of the hypothesis of finite dimensionality in the proof of Theorem 4.3.2. Many of the theorems about adjoints, nevertheless, do not depend on V being finite dimensional. Thus, unless stated otherwise, for the remainder of the discussion we adopt the convention that a reference to the adjoint of a linear operator on an infinite dimensional inner product space assumes its existence.

Theorem 4.3.3 is a useful result for computing adjoints.

<u>Theorem 4.3.3</u> Let V be a finite dimensional inner product space, and let β be an orthonormal basis for V. If T is a linear operator on V, then $[T^*]_\beta = [T]_\beta^*$.

Proof: Let $A = [T]_\beta$, $B = [T^*]_\beta$ and $\beta = \{v_1, v_2, \dots, v_n\}$.

Then from the corollary ~~4~~ 4.2.7 of Theorem 4.2.6, we have

$$b_{ij} = \langle T^*(v_j), v_i\rangle = \overline{\langle v_i, T^*(v_j)\rangle} = \overline{\langle T(v_i), v_j\rangle} = \overline{a_{ji}} = c_{ij}$$

where b_{ij} is the ijth element of B and c_{ij} is the ijth element of A^*. Hence $B = A^*$

Corollary 4.3.4 Let A be an $n\times n$ matrix. Then $L_{A^*} = (L_A)^*$ where L_A is the linear transformation $L_A : F^n \to F^n$ defined by $L_A(x) = Ax$.

Proof: If β is the standard order basis for F^n then by previous result, we have $[L_A]_\beta = A$. Hence $[(L_A)^*]_\beta = [L_A]_\beta^* = A^*$
$= [L_{A^*}]_\beta$

and so $(L_A)^* = L_{A^*}$

As an illustration of Theorem 4.3.3, we compute the adjoint of a specific linear operator.

Example: Let T be the linear operator on $\mathbb{C}^2$ defined by $T(a_1, a_2) = (2ia_1 + 3a_2, a_1 - a_2)$. If β be the standard ordered basis for $\mathbb{C}^2$, then

$$[T]_\beta = \begin{pmatrix} 2i & 3 \\ 1 & -1 \end{pmatrix}$$

So, $$[T^*]_\beta = [T]_\beta^* = \begin{pmatrix} -2i & 1 \\ 3 & -1 \end{pmatrix}$$

Hence $$T^*(a_1, a_2) = (-2ia_1 + a_2, 3a_1 - a_2)$$

The following theorem suggests an analogy between the conjugates of complex numbers and the adjoints of linear operators.

<u>Theorem 4.3.4</u> Let V be an inner product space, and let T and U be linear operators on V. Then

(a) $(T+U)^* = T^* + U^*$;

(b) $(cT)^* = cT^*$ for any $c \in F$;

(c) $(TU)^* = U^*T^*$

(d) $T^{**} = T$

(e) $I^* = I$

Proof: (a) $\langle x, (T+U)^*(y)\rangle = \langle (T+U)(x), y\rangle = \langle T(x)+T(y), y\rangle$

$= \langle T(x), y\rangle + \langle U(x), y\rangle = \langle x, T^*(y)\rangle + \langle x, U^*(y)\rangle$

$= \langle x, T^*(y) + U^*(y)\rangle = \langle x, (T^* + U^*)(y)\rangle$

So, $(T+U)^* = T^* + U^*$

(b) $\langle x, (cT)^*(y)\rangle = \langle (cT)(x), y\rangle$
$= \langle c(T(x)), y\rangle = c\langle T(x), y\rangle = c\langle x, T^*(y)\rangle = \langle x, \bar{c}T^*(y)\rangle$

So, $(cT)^* = \bar{c}T^*$

(c) $\langle x, (TU)^*(y)\rangle = \langle (TU)(x), y\rangle = \langle T(U(x)), y\rangle$
$= \langle U(x), T^*(y)\rangle = \langle x, U^*(T^*(y))\rangle = \langle x, U^*T^*(y)\rangle$

So, $(TU)^* = U^*T^*$

(d) $\langle x, T(y)\rangle = \langle T^*(x), y\rangle = \langle x, T^{**}(y)\rangle$

So, $T^{**} = T$.

(e) $\langle x, I^*(y)\rangle = \langle I(x), y\rangle = \langle x, y\rangle = \langle x, I(y)\rangle$

So, $I^* = I$

Corollary. Let A and B be $n\times n$ matrices. Then

(a) $(A+B)^* = A^* + B^*$

(b) $(cA)^* = \bar{c}A^*$ for all $c \in F$

(c) $(AB)^* = B^*A^*$

(d) $A^{**} = A$

(e) $I^* = I$

Proof: We prove only (c), the remaining parts can be proved similarly and left as exercises.

(c) Since $L_{(AB)^*} = (L_{AB})^* = (L_A L_B)^* = (L_B)^*(L_A)^*$
$= (L_B)^*(L_A)^* = L_{B^*}L_{A^*}$. So, we have $(AB)^* = B^*A^*$

4.4 ~~Bilinear~~ Bilinear and Quadratic forms:

There is a certain class of scalar valued function of two variables defined on a vector space that arises in the study of such diverse subjects as geometry and ~~multi~~ multivariable calculus. This is the class of bilinear forms. We study the basic properties of this class with a special emphasis on symmetric bilinear forms, and we consider some of its applications.

Definition 4.4.1 (~~Bili~~ Bilinear forms) Let V be a vector space over a field F. A function H from the set $V \times V$ of ordered pairs of vectors to F is called a bilinear form on V if H is linear in each variable when the other variable is held fixed; that is, ~~H is bilinear~~ H is a bilinear form on V if

(a) $H(ax_1 + x_2, y) = aH(x_1, y) + H(x_2, y)$ for all $x_1, x_2, y \in V$ and $a \in F$

(b) $H(x, ay_1 + y_2) = aH(x, y_1) + H(x, y_2)$ for all $x, y_1, y_2 \in V$ and $a \in F$

We denote the set of all bilinear forms on V by $\mathcal{B}(V)$. Observe that an inner product on a vector space is a bilinear form if the underlying field is real, but not if the underlying field is complex.

Example 1. Define a function ~~$H: \mathbb{R}^2 \to \mathbb{R}$~~ $H: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$H\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = 2a_1b_1 + 3a_1b_2 + 4a_2b_1 - a_2b_2$$

for $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2$. However, it is more enlightening and less tedious to observe that if $A = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$, $x = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $y = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ then $H(x, y) = x^t A y$. The bilinearity of H now follows directly from the distributive property of matrix multiplication over matrix addition.