The preceding bilinear form is a special case of the next example.

Example 2 Let $V = F^n$, where the vectors are considered as column vectors. For any $A \in M_{n\times n}(F)$, define $H : V\times V \to F$ by $H(x,y) = x^t A y$, for $x, y \in V$. Notice that $H(x,y)$ is a 1×1 matrix. We identify this matrix with its single entry. The bilinearity of H follows as in example 1. For example, for $a \in F$, and $x_1, x_2, y \in V$, we have

$$H(ax_1+x_2, y) = (ax_1+x_2)^t A y = (ax_1^t + x_2^t) A y$$
$$= (ax_1^t) A y + x_2^t A y = a(x_1^t A y) + x_2^t A y = a H(x_1, y) + H(x_2, y)$$

Similarly, we can show $H(x, ay_1+y_2) = a H(x, y_1) + H(x, y_2)$

Now we list several properties possessed by all bilinear forms.

For any bilinear form H on a vector space V over a field F, the following properties hold:

1. If, for any $x \in V$, the functions $L_x, R_x : V \to F$ are defined by $L_x(y) = H(x,y)$ and $R_x(y) = H(y,x)$ for all $y \in V$, then L_x and R_x are linear.

Proof: Let $a \in F$ and $y_1, y_2 \in V$. Then,

$$L_x(ay_1+y_2) = H(x, ay_1+y_2) = aH(x, y_1) + H(x, y_2) \quad (\text{As } H \text{ is bilinear})$$
$$= a L_x(y_1) + L_x(y_2)$$

So, L_x is linear. Similarly, R_x is linear.

2. $H(\theta, x) = H(x, \theta) = 0$ for all $x \in V$

Proof: Exercise.

3. For all $x, y, z, w \in V$,

$$H(x+y, z+w) = H(x,z) + H(x,w) + H(y,z) + H(y,w)$$

Proof: $H(x+y, z+w) = H(x, z+w) + H(y, z+w)$

$= H(x,z) + H(x,w) + H(y,z) + H(y,w)$ (As H is bilinear)

4. If $J: V \times V \to F$ be defined by $J(x,y) = H(y,x)$, then J is a bilinear form.

Proof: Exercise.

Definition 4.4.2. Let V be a vector space, let H_1 and H_2 be two bilinear forms on V, and let a be a scalar. We define the sum $H_1 + H_2$ and the scalar product aH_1 by the equations $(H_1+H_2)(x,y) = H_1(x,y) + H_2(x,y)$ and

$$(aH_1)(x,y) = a(H_1(x,y)) \quad \text{for all } x, y \in V$$

The following theorem is an immediate consequence of the definition:

Theorem 4.4.3 For any vector space V, the sum of two bilinear forms and the product of a scalar and a bilinear form on V are again bilinear forms on V. Furthermore the set $\mathcal{B}(V) = \{H : H \text{ is a } \cancel{\text{bili}} \text{ bilinear form on } V\}$ is a vector space with respect to these operations.

Proof: Exercise.

Let $\beta = \{v_1, v_2, \dots, v_n\}$ be an order basis for an n-dimensional vector space V, let $H \in \mathcal{B}(V)$. We can associate with H an $n \times n$ matrix $A = (a_{ij})_{n \times n}$ where $a_{ij} = H(v_i, v_j)$, $i, j = 1, 2, \dots, n$

Definition 4.4.4 The matrix A defined earlier is called the matrix representation of H with respect to the ordered basis β and is denoted by $\psi_\beta(H)$.

We can therefore regard ψ_β as a mapping from $\mathcal{B}(V)$ to $M_{n\times n}(F)$, where F is the field of scalars for V, that takes a bilinear form H into its matrix representation $\psi_\beta(H)$.

We first consider an example and then show that ψ_β is an isomorphism

Example 3: Consider the bilinear form H of Example 1 (Page-52), and let $\beta = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$ and $B = \psi_\beta(H) = [b_{ij}]_{2\times 2}$

Then
$$b_{11} = H\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right) = 2+3+4-1 = 8$$
$$b_{12} = H\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right) = 2-3+4+1 = 4$$
$$b_{21} = H\left(\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right) = 2+3-4+1 = 2$$
$$b_{22} = H\left(\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right) = 2-3-4-1 = -6$$

So, $\psi_\beta(H) = \begin{pmatrix}8 & 4\\ 2 & -6\end{pmatrix}$

If γ be the standard ordered basis for $\mathbb{R}^2$, the student can verify that $\psi_\gamma(H) = \begin{pmatrix}2 & 3\\ 4 & -1\end{pmatrix}$

Theorem 4.4.5 For any n-dimensional vector space V over F and any ordered basis β for V, ψ_β is an isomorphism.

Proof: Let $\beta = \{v_1, v_2, \ldots, v_n\}$. Let $H_1, H_2 \in \mathcal{B}(V)$ and $a \in F$. Then $\psi_\beta(aH_1 + H_2) = A$ where $a_{ij} = (aH_1 + H_2)(v_i, v_j)$

So, $a_{ij} = (aH_1 + H_2)(v_i, v_j) = a(H_1(v_i, v_j)) + H_2(v_i, v_j)$

So, $a_{ij} = ab_{ij} + c_{ij}$ where $\psi_\beta(H_1) = B = (b_{ij})_{n\times n}$ and $\psi_\beta(H_2) = C = (c_{ij})_{n\times n}$

So, $A = aB + C$

So, $\psi_\beta(aH_1 + H_2) = a\psi_\beta(H_1) + \psi_\beta(H_2)$

So, ψ_β is linear.

To show ψ_β is one-to-one or injective, suppose that $\psi_\beta(H) = O_{n\times n}$ where $O_{n\times n}$ is the null matrix of order n for some $H \in \mathcal{B}(V)$. Fix $v_i \in \beta$ and recall the mapping $L_{v_i} : V \to F$ is linear by property 1 of page-53. By hypothesis $L_{v_i}(v_j) = H(v_i, v_j) = 0$ for all $v_j \in \beta$. Hence L_{v_i} is the zero linear transformation from V to F. So,

$H(v_i, x) = L_{v_i}(x) = 0$ for all $x \in V$ and $v_i \in \beta$. ...(1)

Next fix an arbitrary $y \in V$, and recall the linear mapping $R_y : V \to F$ defined in the property 1 on page-53

By (1), $R_y(v_i) = H(v_i, y) = 0$ for all $v_i \in \beta$ and hence R_y is the zero linear transformation,

So, $H(x, y) = R_y(x) = 0$ for all $x, y \in V$. So H is the zero bilinear form, and so ψ_β is one-to-one.

To show that ψ_β is onto, consider any $A \in M_{n\times n}(F)$

Now, we recall the isomorphism $\phi_\beta : V \to F^n$ defined by $\phi_\beta(x) = [x]_\beta$ where $[x]_\beta$ is the