and $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = s_1 x_1 + s_2 x_2$, then $f(t_1, t_2) = \lambda_1 s_1^2 + \lambda_2 s_2^2$. We can think s_1 and s_2 as the coordinates of (t_1, t_2) relative to γ. Thus the polynomial $f(t_1, t_2)$, as an expression involving the coordinates of points with respect to standard ordered basis for $\mathbb{R}^2$, is transformed into new polynomial $g(s_1, s_2) = \lambda_1 s_1^2 + \lambda_2 s_2^2$ interpreted as an expression involving the coordinates of a point relative to the new ordered basis γ.

Let H denote the symmetric bilinear form corresponding to the quadratic form defined by (1), let β be the standard ordered basis for $\mathbb{R}^2$, and let $A = \psi_\beta(H)$

Then $A = \psi_\beta(H) = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$ as $H(e_1, e_1) = \frac{1}{2}\left[K(2e_1) - K(e_1) - K(e_1)\right] = 5$ etc.

Next, we find an orthogonal matrix Q such that $Q^t A Q$ is a diagonal matrix. For this purpose, observe that $\lambda_1 = 6$ and $\lambda_2 = 1$ are the eigenvalues of A with corresponding orthonormal eigenvectors

$$v_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } v_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Let $\gamma = \{v_1, v_2\}$. Then γ is an orthonormal basis for $\mathbb{R}^2$ ~~com~~ consisting of eigenvectors of A. Hence, setting $Q = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$, we see that Q is an orthogonal matrix and $Q^t A Q = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$

Clearly, Q is also a change of coordinate matrix.

Consequently, $\psi_\gamma(H) = Q^t \psi_\beta(H) Q = Q^t A Q = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$

Thus, by corollary 4.8.2, ~~there exists~~ $K(x) = 6 s_1^2 + s_2^2$

for any $x = s_1 v_1 + s_2 v_2 \in \mathbb{R}^2$, So, $g(s_1, s_2) = 6s_1^2 + s_2^2$

The next problem shows how the theory of quadratic forms can be applied to the problem of describing quadratic surfaces in $\mathbb{R}^3$

Example 3 Let S be the surface in $\mathbb{R}^3$ defined by the equation

$$2t_1^2 + 6t_1t_2 + 5t_2^2 - 2t_2t_3 + 2t_3^2 + 3t_1 - 2t_2 - t_3 + 14 = 0 \quad \cdots (1)$$

Then (1) describes the points of S in terms of their coordinates relative to β, the standard ordered basis for $\mathbb{R}^3$. We find a new orthonormal basis γ for $\mathbb{R}^3$ so that the equation describing the coordinates of S relative to γ is simpler than (1)

We begin with the observation that the terms of second degree on the left side of (1) add to form a quadratic form K on $\mathbb{R}^3$:

$$K\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = 2t_1^2 + 6t_1t_2 + 5t_2^2 - 2t_2t_3 + 2t_3^2$$

Next, we diagonalize K. Let H be the symmetric Bilinear form corresponding to K, and let $A = \psi_\beta(H)$. Then

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 5 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad (\text{verify it})$$

The characteristic polynomial of A is

$(-1)(t-2)(t-7)t$ (verify it); hence A has the eigenvalues $\lambda_1 = 2$, $\lambda_2 = 7$, $\lambda_3 = 0$ Corresponding unit eigenvectors are

$$v_1 = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{35}}\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} \text{ and } v_3 = \frac{1}{\sqrt{14}}\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \quad (\text{verify it})$$

Set $\gamma = \{v_1, v_2, v_3\}$ and

$$Q = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{35}} & -\frac{3}{\sqrt{14}} \\ 0 & \frac{5}{\sqrt{35}} & \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{35}} & \frac{1}{\sqrt{14}} \end{pmatrix}$$

As in example 1, Q is the change of coordinate matrix changing γ-coordinates to β-coordinates, and

$$\psi_\gamma(H) = Q^t \psi_\beta(H) Q = Q^t A Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

By the corollary 4.8.2 of Theorem 4.8.1, if $x = s_1 v_1 + s_2 v_2 + s_3 v_3$, then $K(x) = 2s_1^2 + 7s_2^2 \quad \ldots (2)$

We are now ready to transform (1) into an equation involving coordinates relative to γ. Let $x = (t_1, t_2, t_3) \in \mathbb{R}^3$ and suppose that $x = s_1 v_1 + s_2 v_2 + s_3 v_3$. Then by our previous result,

$$x = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = Q \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

and so,

$$t_1 = \frac{s_1}{\sqrt{10}} + \frac{3}{\sqrt{35}} s_2 - \frac{3}{\sqrt{14}} s_3$$

$$t_2 = \frac{5 s_2}{\sqrt{35}} + \frac{2}{\sqrt{14}} s_3$$

$$t_3 = \frac{3 s_1}{\sqrt{10}} - \frac{s_2}{\sqrt{35}} + \frac{s_3}{\sqrt{14}}$$

Thus, $3t_1 - 2t_2 - t_3 = -\frac{14 s_3}{\sqrt{14}} = -\sqrt{14}\, s_3 \quad \ldots (3)$

Combining (1), (2) & (3) we conclude that if $x \in \mathbb{R}^3$

and $x = s_1 v_1 + s_2 v_2 + s_3 v_3$ then $x \in S$ if and only if

$$2s_1^2 + 7s_2^2 - \sqrt{14}\, s_3 + 14 = 0 \text{ or, } s_3 = \frac{2}{\sqrt{14}} s_1^2 + \frac{7}{\sqrt{14}} s_2^2 + \sqrt{14} = s_3$$

Consequently, if we draw new axes OX', OY', OZ' along v_1, v_2, v_3 and the new ~~coord~~ coordinates become (x', y', z') and the equation becomes $z' = \frac{2}{\sqrt{14}}(x')^2 + \frac{7}{\sqrt{14}}(y')^2 + \sqrt{14}$, So, that the graph of the equation coincides with the surface S, we recognize now S to be an elliptic paraboloid.

4.9 Second derivative test for critical point of a function of several variables.

We now consider an application of theory of quadratic forms to multivariate calculus – the derivation of the second derivative test for local extrema of a function of several variables. Here we assume that the students are acquainted with the calculus of function of several variables upto Taylor's theorem.

Let $z = f(t_1, t_2, \ldots, t_n)$ be a fixed real valued function of n real variables for which all third order partial derivatives exist and are continuous. The function f is said to have a ~~local~~ local maximum at a $p \in \mathbb{R}^n$ if there exists a $\delta > 0$ such that $f(p) \geq f(x)$ whenever $\|x - p\| < \delta$. Likewise, f has a local minimum at $p \in \mathbb{R}^n$ if $\exists$ a $\delta > 0$ such that $f(p) \leq f(x)$ whenever $\|x - p\| < \delta$. If f has either a local minimum or ~~max~~ a local maximum at p, we say that f has a local extremum at p. A point $p \in \mathbb{R}^n$ is called a critical point of f if $\frac{\partial f(p)}{\partial t_i} = 0$ for $i = 1, 2, \ldots, n$. It is a well known fact that