co-ordinate vector of x relative to β. for all $x \in V$, i.e., if for $x \in V$ $x = \sum_{i=1}^{n} a_i v_i$ then $[x]_\beta$ written as a column is $[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$. So, $\phi_\beta(x) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

Let $H : V \times V \to F$ be the mapping defined by

$$H(x,y) = \{\phi_\beta(x)\}^t A \{\phi_\beta(y)\} \quad \text{for all } x, y \in V$$

Then we can prove That $H \in \mathcal{B}(V)$ (Exercise).

we show that $\psi_\beta(H) = A = (a_{ij})_{n\times n}$. Let $v_i, v_j \in \beta$

Then $\phi_\beta(v_i) = e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \to$ ith component is 1

Similarly, $\phi_\beta(v_j) = e_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \to$ jth component is 1

Hence, for any i and j

$$H(v_i, v_j) = \{\phi_\beta(v_i)\}^t A \{\phi_\beta(v_j)\} = e_i^t A e_j = a_{ij}$$

So, we conclude that $\psi_\beta(H) = A$. So ψ_β is onto

Hence ψ_β is an isomorphism.

Corollary 4.4.6 For any n-dimensional vector space, V, $\mathcal{B}(V)$ has dimension n^2

Proof: As $\mathcal{B}(V)$ is isomorphic to $M_{n\times n}(F)$ and $M_{n\times n}(F)$ has dimension n^2. So, $\mathcal{B}(V)$ has dimension n^2.

The following corollary is easily established (prove it) by reviewing the proof of 4.4.5.

Corollary 4.4.7 Let V be an n-dimensional vector space over a field F with ordered basis β. If $H \in \mathcal{B}(V)$ and $A \in M_{n\times n}(F)$, then $\psi_\beta(H) = A$ if and only if

$$H(x,y) = \{\phi_\beta(x)\}^t A \{\phi_\beta(y)\} \quad \text{for all } x, y \in V$$

The following result is an immediate consequence of corollary 4.4.7 (Prove it.):

Corollary 4.4.8. Let F be a field, ~~and~~ n a positive integer and β be the standard ordered basis for F^n. Then for and $H \in \mathcal{B}(F^n)$, $\exists$ a unique matrix $A \in M_{n\times n}(F)$, namely, $A = \psi_\beta(H)$ such that

$$H(x,y) = x^t A y \quad \text{for all } x, y \in F^n$$

Example 4 Define a function $H: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$H = \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = \det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = a_1 b_2 - a_2 b_1 \quad \text{for}$$

$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2$

It can be shown (check it) that H is a bilinear form. We find the matrix A in corollary 4.4.8 such that $H(x,y) = x^t A y$ for all $x, y \in \mathbb{R}^2$

Since $a_{ij} = H(e_i, e_j)$ for all i and j, a_{ij} the ijth element of A. we have $a_{11} = \det\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 0$ $a_{12} = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$

$$a_{21} = \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1 \qquad a_{22} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = 0$$

So, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

There is an analogy between bilinear forms and linear operators on finite dimensional vector spaces in that both are associated with unique square matrices and the correspondences depends on the choice of an ordered basis for the vector space. As in the case of linear operators, one can pose the following question: How does the matrix ~~corres~~ ~~corres~~ corresponding to a fixed bilinear form change when the ordered basis is changed? As we have seen, the corresponding question for matrix representations of linear operators leads to the definition of similarity relation on square matrices. In the case of bilinear forms, the corresponding question leads to another relation on square matrices, the congruence relation.

Definition: Let $A, B \in M_{n\times n}(F)$. Then B is said to be congruent to A if there exists an invertible matrix $Q \in M_{n\times n}(F)$ such that $B = Q^t A Q$. Observe that relation of congruence is an equivalence relation (Prove it)

The next Theorem relates congruence to the matrix representation of a bilinear form.

Theorem 4.4.9 Let V be a finite dimensional vector space with ordered bases $\beta = \{v_1, v_2, \dots, v_n\}$ and $\gamma = \{w_1, w_2, \dots, w_n\}$ and let Q be the change of coordinate matrix changing γ-coordinates into β-coordinates. Then for any $H \in \mathcal{B}(V)$, we have $\Psi_\gamma(H) = Q^t \Psi_\beta(H) Q$. Therefore $\Psi_\gamma(H)$ is congruent to $\Psi_\beta(H)$.

Proof: Suppose that $A = [a_{ij}]_{n\times n} = \Psi_\beta(H)$ and $B = [b_{ij}]_{n\times n} = \Psi_\gamma(H)$

Then for $1 \le i, j \le n$

Then $w_i = \sum_{k=1}^{n} q_{ki} v_k$ and $w_j = \sum_{r=1}^{n} q_{rj} v_r$. Where $Q = [q_{ij}]_{n\times n}$

Then
$$
\begin{aligned}
b_{ij} &= H(w_i, w_j) = H\left(\sum_{k=1}^{n} q_{ki} v_k, w_j\right) \\
&= \sum_{k=1}^{n} q_{ki} H(v_k, w_j) \\
&= \sum_{k=1}^{n} q_{ki} H\left(v_k, \sum_{r=1}^{n} q_{rj} v_r\right) \\
&= \sum_{k=1}^{n} q_{ki} \sum_{r=1}^{n} q_{rj} H(v_k, v_r) \\
&= \sum_{k=1}^{n} q_{ki} \sum_{r=1}^{n} q_{rj} a_{kr} \\
&= \sum_{k=1}^{n} q_{ki} c_{kj} \quad \text{where } AQ = [c_{ij}]_{n\times n} \\
&= \sum_{k=1}^{n} d_{ik} c_{kj} \quad \text{where } Q^t = [d_{ij}]_{n\times n} \\
&= f_{ij} \quad \text{where } Q^t A Q = [f_{ij}]_{n\times n}
\end{aligned}
$$

Hence $B = Q^t A Q$