The following result is the converse of Theorem 4.4.9

Corollary 4.4.10 Let V be an n-dimensional vector space with ordered basis β and let H be a bilinear form on V. For any $n\times n$ matrix B, if B is congruent to $\psi_\beta(H)$, then $\exists$ an ordered basis γ for V such that $\psi_\gamma(H) = B$. Furthermore, if $B = Q^t \psi_\beta(H) Q$ for some invertible matrix Q, then Q changes γ-coordinate into β-coordinates.

Proof: Suppose that $B = Q^t \psi_\beta Q$ for some invertible matrix Q and that $\beta = \{v_1, v_2, \ldots, v_n\}$. Let $\gamma = \{w_1, w_2, \ldots, w_n\}$, where

$$w_j = \sum_{i=1}^{n} q_{ij} v_i \quad \text{for } 1 \le j \le n, \text{ and } Q = [q_{ij}]_{n\times n}.$$

Since Q is invertible, γ is an ordered basis for V, and Q is the change of coordinate matrix that changes γ-coordinates into β-coordinates. So, by Theorem 4.4.5,

$$B = Q^t \psi_\beta(H) Q = \psi_\gamma(H)$$

4.5 Symmetric Bilinear Forms

Like the diagonalization problem for linear operators, there is an ~~also~~ analogous diagonalization problem for bilinear forms, namely, the problem of determining those bilinear forms for which there are diagonal matrix representations. We will see that there is a close relationship between diagonalizable bilinear forms and those that are called symmetric.

Definition 4.5.1 A bilinear form H on a vector space V is symmetric if $H(x,y) = H(y,x)$ for all $x, y \in V$

As the name suggests, symmetric bilinear forms correspond to symmetric matrices.

Theorem 4.5.2 Let H be a bilinear form on a finite dimensional vector space V, and let β be an ordered basis for V. Then H is symmetric if and only if $\psi_\beta(H)$ is symmetric.

Proof: Let $\beta = \{v_1, v_2, \ldots, v_n\}$ and $B = \psi_\beta(H)$

First assume that H is symmetric. Let $B = (b_{ij})_{n\times n}$

Then for $1 \le i, j \le n$,

$$b_{ij} = H(v_i, v_j) = H(v_j, v_i) = b_{ji},$$

and it follows that B is symmetric.

Conversely, suppose that B is symmetric. Let F be the field of scalars for V. Let $J : V\times V \to F$ be the mapping defined by $J(x,y) = H(y,x)$ for all $x, y \in V$. By Property 4 of Page-54, J is a bilinear form. Let $C = (c_{ij})_{n\times n} = \psi_\beta(J)$

Then, for $1 \le i, j \le n$,

$$c_{ij} = J(v_i, v_j) = H(v_j, v_i) = b_{ji} = b_{ij}$$ where

$B = (b_{ij})_{n\times n}$ and B is symmetric.

So, $C = B$. Since ψ_β is one-to-one, we have $J = H$.

Hence $H(y,x) = J(x,y) = H(x,y)$ for all $x, y \in V$. So, H is symmetric.

Definition 4.5.3 A bilinear form H on a finite dimensional vector space V is called diagonalizable if $\exists$ an ordered basis β for V such that $\psi_\beta(H)$ is a diagonal matrix.

Corollary 4.5.4. Let H be a diagonalizable bilinear form on a finite dimensional vector space V. Then H is symmetric.

Proof: Suppose that H is diagonalizable. Then $\exists$ an ordered basis β for V such that $\psi_\beta(H) = D$, where D is a diagonal matrix. As ~~is~~ and D is a diagonal matrix, it is symmetric. So, by Theorem 4.5.2, H is symmetric.

Unfortunately, the converse is true. It is illustrated by the following example 1: Let $F = \mathbb{Z}_2$, $V = F^2$ and $H : V \times V \to F$ be the bilinear form defined by

$$H\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = a_1 b_2 + a_2 b_1$$

So, $\cancel{H\left(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right) = a_2 b_1 + a_1 b_2 = a_1 b_2 + a_2 b_1 = H\left(\begin{pmatrix} a_1 \\ \end{pmatrix}\right.}$

So, $H\left(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right) = b_1 a_2 + b_2 a_1 = a_1 b_2 + a_2 b_1$ as F is a field.

So, $H\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = H\left(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right)$ for all $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in V$

So, H is symmetric. Let β be the standard ordered basis of V. If $\psi_\beta(H) = A = (a_{ij})_{2\times 2}$

Then $a_{11} = H\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = 0$, $a_{12} = H\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = 1$

$a_{21} = H\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = 1$ $\quad a_{22} = H\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = 0$

So, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, a symmetric matrix. We show that H is not diagonalizable.

~~By~~ If possible, let H be diagonalizable. Then there is an ordered basis γ for V ~~for which~~ such that

$B = \psi_\gamma(H)$ is a diagonal matrix. So by Theorem 4.4.9, $\exists$ an invertible matrix Q such that $B = Q^t A Q$. Since Q is invertible, it follows that $\text{rank}(B) = \text{rank}(A) = 2$. Consequently, the diagonal entries of B are non-zero (as B is a diagonal matrix). Since the only non-zero scalar of F is 1, so,
$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

~~Suppose~~ Suppose that $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then
$$B = Q^t A Q = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$= \begin{pmatrix} c & a \\ d & b \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ac+ac & bc+ad \\ bc+ad & bd+bd \end{pmatrix}$$

But $p + p = 0$ for all $p \in F$; hence $ac + ac = 0$. So, comparing row 1, column 1 entries of the matrices, in the equation above, we conclude that $1 = 0$, a contradiction. So, H is not diagonalizable.

The failure of the bilinear form in the above example to be diagonalizable is due to the fact that the scalar field $\mathbb{Z}_2$ is of characteristic 2. Recall that a field F is of characteristic 2 if $1+1 = 0$ in F. If F is not of characteristic 2, then $1+1$ is a non-zero element and it has a multiplicative inverse.

Before proving the converse of corollary 4.5.4 of Theorem 4.5.2 for scalar fields that are not of characteristic 2, we establish the following lemma:

Lemma 4.5.5 Let H be a non-zero symmetric bilinear form on a vector space V over a field F which is not characteristic two. Then $\exists$ a vector $x \in V$ such that $H(x, x) \neq 0$.