Proof: Since H is non-zero, we choose vector $u, v \in V$ such that $H(u,v) \neq 0$. If $H(u,u) \neq 0$ or $H(v,v) \neq 0$ there is nothing to prove. Otherwise, set $x = u+v$. Then

$$\begin{aligned} H(x,x) &= H(u+v, u+v) = H(u,u) + H(u,v) + H(v,u) + H(v,v) \\ &= 2H(u,v) \quad (\text{As } H(u,u) = H(v,v) = 0 \text{ and } H \text{ is symmetric}) \\ &\neq 0 \quad \text{as } 2 \neq 0 \text{ and } H(u,v) \neq 0 \text{ \& } F \text{ is field of characteristic} \neq 2.) \end{aligned}$$

Theorem 4.5.6 Let V be a finite dimensional vector space over a field F not of characteristic two. Then every symmetric bilinear form is diagonalizable

Proof: We use mathematical induction on $n = \dim(V)$. If $n=1$, then every element of $\mathcal{B}(V)$ is diagonalizable. Now suppose that theorem is valid vector space of dimension less than n for some fixed integer $n > 1$, and suppose that $\dim(V) = n$. If H is the zero bilinear form on V, then trivially it is diagonalizable; so suppose that H is a non-zero symmetric bilinear form on V. By the Lemma 4.5.5, $\exists$ a non-zero vector $x \in V$ such that $H(x,x) \neq 0$. Recall the function $L_x : V \to F$ defined by $L_x(y) = H(x,y)$ for all $y \in V$. By property 1 on Page-53, L_x is linear. Furthermore, since $L_x(x) = H(x,x) \neq 0$, L_x is non-zero. Consequently, $\operatorname{rank}(L_x) = 1$ and hence $\dim(N(L_x)) = n-1$ where $\operatorname{rank}(L_x)$ = dimension of the image space $\{L_x(y) : y \in V\}$ and $\dim(N(L_x))$ = dimension of the null space $N(L_x) = \{y \in V : L_x(y) = 0\}$

Now, the restriction of H to $N(L_x)$ is also a symmetric bilinear form on a vector space of dimension. So, by Induction hypothesis, $\exists$ an ordered basis $\{v_1, v_2, \ldots, v_{n-1}\}$ for $N(L_x)$ such that

$H(v_i, v_j) = 0$ for $i \neq j$, $1 \le i, j \le n-1$. Set $v_n = x$. Then $v_n \notin N(L_x)$ and so $\beta = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$ is an ordered basis for V. In addition, $H(v_i, v_n) = H(v_n, v_i) = 0$ for $i = 1, 2, \ldots, n-1$. So, we conclude that $\psi_\beta(H)$ is a diagonal matrix and therefore H is diagonalizable.

Corollary 4.5.7 Let F be a field that is not of characteristic two. If $A \in M_{n\times n}(F)$ is a symmetric matrix, then A is congruent to a diagonal matrix.

Proof: Exercise

4.6. Diagonalization of Symmetric matrices

Let A be a symmetric $n\times n$ matrix with entries from a field F not of characteristic two. By the corollary 4.5.7 of Theorem 4.5.6, there are matrices $Q, D \in M_{n\times n}(F)$ such that Q is invertible, D is diagonal and $Q^tAQ = D$. We ~~can~~ now give a method of computing Q and D. This method requires familiarity with elementary matrices and their properties. You can review it as it has been done earlier.

If E is an elementary $n \times n$ matrix, Then AE can be ~~performed~~ obtained by performing an elementary column operation on A. Also E^tA can be obtained by performing the same operation on the rows of A. Thus ~~E^tA~~ E^tAE can be obtained from by performing an elementary ~~column~~ operation on ~~A~~ ~~on~~ the columns of A and then performing the same operation on the rows of AE (Note that the order of operations can be reversed because of the associative property of matrix multiplication)

Suppose that Q is an invertible matrix and D is a diagonal matrix such that $Q^t A Q = D$, we know from our previous result on ~~matrices~~ elementary matrices, Q can be written as product of finite number of elementary matrices, say $Q = E_1 E_2 \cdots E_k$ So,

$$D = Q^t A Q = E_k^t E_{k-1}^t \cdots E_1^t A E_1 E_2 \cdots E_k$$

From the preceding equation, we conclude that by means of several elementary column operations and the corresponding row operations, A can be transformed into a diagonal matrix D. Furthermore, if $E_1, E_2, \ldots, E_k$ are the elementary matrices corresponding to these elementary column operations indexed in the order performed, and if $Q = E_1 E_2 \cdots E_k$, then $Q^t A Q = D$.

Example 1 Let A be the symmetric matrix in $M_{3\times 3}(\mathbb{R})$ defined by
$$A = \begin{pmatrix} 1 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

We use the procedure just described to find an invertible matrix Q and a diagonal matrix D such that $Q^t A Q = D$.

We begin by eliminating all the non-zero ~~of~~ entries in the first row and first column except for the entry in the column 1 and row 1. To do this, we add the first of A to the second column to produce a zero in row 1 and column 2. The elementary matrix that corresponds to this elementary column operation is

$$E_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we perform the corresponding elementary row operations on the rows of AE_1 to obtain $E_1^t A E_1 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$

we now use the first column of $E_1^t A E_1$ to eliminate the 3 in row 1 column 3, and follow this operation with corresponding row operation. The corresponding elementary matrix E_2 and the result of the elementary operations $E_2^t E_1^t A E_1 E_2$ are, respectively,

$$E_2 = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } E_2^t E_1^t A E_1 E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 4 & -8 \end{pmatrix}$$

Finally we subtract 4 times the second column of $E_2^t E_1^t A E_1 E_2$ from the third column and follow this with corresponding row operation. The corresponding elementary matrix E_3 and the result of the elementary operations $E_3^t E_2^t E_1^t A E_1 E_2 E_3$ are, respectively,

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } E_3^t E_2^t E_1^t A E_1 E_2 E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -24 \end{pmatrix}$$

So, we have obtained a diagonal matrix, the process is complete. So we let $Q = E_1 E_2 E_3 = \begin{pmatrix} 1 & 1 & -7 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$

and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -24 \end{pmatrix}$ to obtain the

derived diagonalization $Q^t A Q = D$.

The reader should justify the following method of computing Q without recording each elementary