matrix separately (but can write the short form of elementary column operations and row operations that we have used previously). The method is inspired by the algorithm for computing the inverse of a matrix. We use a sequence of elementary column operations and corresponding row operations to change the $n \times 2n$ matrix (A/I), (I is the identity matrix of order n) into the form (D/B) where D is a diagonal matrix and $B = Q^t$. It then follows that $D = Q^t A Q$.

Starting with the matrix A of the preceding example, this method produces the following sequence of matrices:

$$(A/I) = \left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{C_2+C_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 3 & 4 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2+R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 & 0 \\ 3 & 4 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{C_3-3C_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 & 0 \\ 3 & 4 & -8 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3-3R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 & 0 \\ 0 & 4 & -8 & -3 & 0 & 1 \end{array}\right) \xrightarrow{C_3-4C_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 4 & -24 & -3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3-4R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -24 & -7 & -4 & 1 \end{array}\right) = (D \mid Q^t)$$

So, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -24 \end{pmatrix}$, $Q^t = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -7 & -4 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 & -7 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$

4.7 Quadratic forms

Associated with symmetric bilinear forms are functions called Quadratic forms.

Definition 1.7.1: Let V be a vector space over F. A function $K : V \to F$ is called a quadratic form if $\exists$ a symmetric bilinear form $H \in \mathcal{B}(V)$ such that $K(x) = H(x,x)$ for all $x \in V$ $\quad \cdots (1)$

If the field F is not of characteristic two, there is a one-to-one correspondence between symmetric bilinear forms and quadratic forms given by (1). In fact, if K be a quadratic form on a vector space V over a field F not of characteristic two, $K(x) = H(x,x)$ for some symmetric bilinear form H on V, then we can recover H from K because

$$H(x,y) = \frac{1}{2}\left[K(x+y) - K(x) - K(y)\right] \quad \cdots \quad (2)$$

Example 1

The classic example of a quadratic form is the homogeneous second degree polynomials of several variables. Given the variables $t_1, t_2, \ldots, t_n$ that takes values in a field F not of characteristic two ~~&~~ and given (not necessarily distinct) scalars a_{ij}, ~~$1 \le i \le j \le n$~~ $(1 \le i \le j \le n)$, define the polynomial

$$f(t_1, t_2, \ldots, t_n) = \sum_{i \le j} a_{ij} t_i t_j$$

Any such polynomial is a quadratic form. In fact, if β is the standard order basis for F^n, then the symmetric bilinear form H ~~corresponding~~ corresponding to the quadratic form f has the matrix representation $\psi_\beta(H) = ~~A~~ B$ ~~where~~ where ~~$A = (a_{ij})_{n\times n}$ and $a_{ij} = a_{ji}$~~ where $B = (b_{ij})_{n \times n}$ and

$$b_{ij} = b_{ji} = \begin{cases} a_{ii} & i = j \\ \frac{1}{2} a_{ij} & i \ne j \end{cases}$$

To see this, apply (2) to obtain $H(e_i, e_j) = b_{ij}$ from the quadratic form K, and verify that f is computable from H by (1) using f

in place of K.

For example, given the polynomial (in three variables t_1, t_2, t_3)

$$f(t_1,t_2,t_3) = 2t_1^2 - t_2^2 + 6t_1t_2 - 4t_2t_3$$ with real coefficients,

let $$B = \begin{pmatrix} 2 & 3 & 0 \\ 3 & -1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$$

Setting $H(x,y) = x^t B y$ for all $x, y \in \mathbb{R}^3$, we see that

$$f(t_1,t_2,t_3) = (t_1,t_2,t_3)\, A \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \quad \text{for} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \in \mathbb{R}^3$$

4.8 Quadratic forms over the field $\mathbb{R}$.

It can be shown that symmetric matrices over $\mathbb{R}$ are orthogonally diagonalizable (A is orthogonally diagonalizable if $\exists$ an orthogonal matrix P such that $P^tAP = D$, where D is a diagonal matrix).

So, the theory of symmetric bilinear forms and quadratic forms on finite dimensional vector space over $\mathbb{R}$ is especially nice. The following theorem and its corollary are useful

Theorem 4.8.1 Let V be a finite dimensional real inner product space and let H be a symmetric bilinear form on V. Then $\exists$ an orthonormal basis β for V such that $\psi_\beta(H)$ is a diagonal matrix.

Proof: Choose any orthonormal basis $\gamma = \{v_1, v_2, \dots, v_n\}$ for V, and let $A = \psi_\gamma(H)$. Since A is symmetric, $\exists$ an orthogonal matrix Q and a diagonal matrix D such that $D = Q^tAQ$. Let $\beta = \{w_1, w_2, \dots, w_n\}$ be defined by

$$w_j = \sum_{i=1}^{n} q_{ij} v_i \quad \text{for } 1 \le j \le n \quad \text{where } Q = (q_{ij})_{n\times n}$$

By Theorem 4.4.9, $\psi_\beta(H) = D$. Furthermore, since Q is orthogonal and γ is orthonormal, β is orthonormal (verify it).

Corollary 4.8.2 Let K be a quadratic form on a finite dimensional real inner product space V. There exists an orthonormal basis $\beta = \{v_1, v_2, \ldots, v_n\}$ for V and scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct) such that if $x \in V$

and $$x = \sum_{i=1}^{n} s_i v_i \quad , s_i \in \mathbb{R}, \ 1 \le i \le n$$

then $$K(x) = \sum_{i=1}^{n} \lambda_i s_i^2$$

In fact, if H is the symmetric bilinear form determined by K, then β can be chosen to be any orthonormal basis for V such that $\psi_\beta(H)$ is a diagonal matrix.

Proof: Let H be a symmetric bilinear form for which $K(x) = H(x,x)$ for all $x \in V$. By Theorem 4.8.1, $\exists$ an orthonormal basis $\beta = \{v_1, v_2, \ldots, v_n\}$ for V such that $\psi_\beta(H)$ is the diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Let $x \in V$ and suppose that $x = \sum_{i=1}^{n} s_i v_i$. Then

$$K(x) = H(x,x) = [\phi_\beta(x)]^t D [\phi_\beta(x)] = (s_1, s_2, \ldots, s_n) D \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} = \sum_{i=1}^{n} \lambda_i s_i^2$$

Example 2 For the homogeneous real polynomial of degree 2 defined $f(t_1, t_2) = 5t_1^2 + 2t_2^2 + 4t_1t_2 \quad \ldots \quad (1)$

we find ~~that~~ an orthonormal basis $\gamma = \{x_1, x_2\}$ for $\mathbb{R}^2$ and scalars λ_1, λ_2 such that if $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in \mathbb{R}^2$