and $\frac{1}{2}\lambda_j + \varepsilon < 0$. Let s be a real number such that $0 < |s| < \delta$. Substituting $x = s v_i$ and $x = s v_j$ into the left inequality and right inequality of (5) respectively, we obtain

$$f(\theta) = 0 < \left(\tfrac{1}{2}\lambda_i - \varepsilon\right)s^2 < f(s v_i) \quad \text{and}$$

$$f(s v_j) < \left(\tfrac{1}{2}\lambda_j + \varepsilon\right)s^2 < 0 = f(\theta)$$

So, f attains both positive and negative values arbitrarily close to θ. So f has neither a local maximum or a local minimum. This establishes (c).

To show that the second derivative test is inconclusive under the condition stated in (d), consider the function

$$f(t_1, t_2) = t_1^2 - t_2^4 \quad \text{and} \quad g(t_1, t_2) = t_1^2 + t_2^4$$

at $p = \theta = (0,0)$. In both cases, the function has a critical point at p and

$$A(p) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

However f does not have a local extremum at θ, whereas g has a local minimum at θ.

7.10 Sylvester's law of Inertia

Any two matrix representations of a bilinear form have the same rank because the rank is preserved under congruence. We can therefore define the rank of a bilinear form to be the rank of any of its matrix representations. If a matrix representation is a diagonal matrix, then the rank is equal to the number of non-zero

diagonal entries of the matrix.
We confine our analysis to symmetric bilinear forms on finite dimensional real vector spaces. Each such form has a diagonal matrix representation in which the diagonal entries may be positive, negative, or zero. ~~Also these~~ Although these entries are not unique, we show that the number of entries that are positive and the number that are negative are unique. That is, they are independent of the choice of diagonal representation. This result is called Sylvester's law of inertia. We prove the law and apply it to describe the equivalent classes of congruent symmetric real matrices.

Theorem 4.10.1 (Sylvester's Law of Inertia). Let H be a symmetric bilinear form on a finite dimensional real vector space V. Then the number of positive diagonal entries and the number of negative diagonal entries in any diagonal matrix representation of H are each independent of the diagonal representation.

Proof: Suppose that β and γ are ordered bases for V that determine diagonal representations of H. Without loss of generality, we may assume that β and γ are ordered so that on each diagonal the entries are in the order of positive, negative, and zero. It suffices to show that both representations have the same number of positive entries because the number of negative entries is equal to the difference between the rank and the number of positive entries. Let p and q be the number of

~~positive entries~~ diagonal entries in the matrix representations of H with respect to β and γ, respectively. We suppose that $p \neq q$ and arrive at a contradiction. Without loss of generality, let $p < q$. ~~Let~~ $\cancel{\beta = \{v_1, v_2, \ldots, v_p, \ldots, v_n\}}$

~~and~~ $\cancel{\gamma = \{w_1, w_2, \ldots, w_q, \ldots, w_n\}}$, ~~where~~

Let $\beta = \{v_1, v_2, \ldots v_p, \cdots v_r, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots w_q, \cdots w_r, \cdots w_n\}$ where r is the rank of H and $n = \dim(V)$.

Let $L : V \to \mathbb{R}^{p+r-q}$ be the mapping defined by

$$L(x) = \left(H(x, v_1), H(x, v_2), \ldots H(x, v_p), H(x, w_{q+1}), \ldots, H(x, w_r)\right)$$

It is easily verified (verify it) that L is linear and $\operatorname{rank}(L) \leq p+r-q$. Hence $\operatorname{nullity}(L) \geq n-(p+r-q) > n-r$.

So, $\exists$ a non-zero vector $v_0 \notin \operatorname{span}(\{v_{r+1}, v_{r+2}, \ldots v_n\})$, but ~~$v_0 \in N(L)$ (null space~~ $v_0 \in N(L)$ (null space of L). Since $v_0 \in N(L)$, it follows that $H(v_0, v_i) = 0$ for $i \leq p$ and $H(v_0, w_i) = 0$ for $q < i \leq r$. Suppose that

$$v_0 = \sum_{j=1}^{n} a_j v_j = \sum_{j=1}^{n} b_j w_j .$$

For any $i \leq p$,

$$H(v_0, v_i) = H\left(\sum_{j=1}^{n} a_j v_j, v_i\right) = \sum_{j=1}^{n} H(v_j, v_i) = a_i H(v_i, v_i)$$

But for $i \leq p$, we have $H(v_i, v_i) > 0$. So, that $a_i = 0$.

So, $a_i = 0$ for $i \leq p$. Similarly, $b_i = 0$ for $q+1 \leq i \leq r$.

Since v_0 is not in the span of $\{v_{r+1}, v_{r+2}, \ldots v_n\}$, it follows that $a_i \neq 0$ for some $p < i \leq r$. ~~Thus~~

$\cancel{H(v_0, v_0) = H\left(\sum_{j=1}^{n} b_j v_j, \sum_{j=1}^{n} b_j v_j\right)}$ $\cancel{H(v_0, v_0) = H\left(\sum_{j=1}^{n} b_j w_j, \sum_{i=1}^{n} b_i w_i\right)}$

$\cancel{= \sum_{j=1}^{n} b_{ij}^2 H(w_j, w_j)}$ Thus, we have,

$$H(v_0, v_0) = H\left(\sum_{j=1}^{n} a_j v_j, \sum_{i=1}^{n} a_i v_i\right) = \sum_{j=1}^{n} a_j^2 H(v_j, v_j) = \sum_{j=p+1}^{r} a_j^2 H(v_j, v_j) < 0$$

and
$$H(v_0, v_0) = H\left(\sum_{j=1}^{n} b_{j} w_j, \sum_{i=1}^{n} b_{i} w_i\right) = \sum_{j=1}^{n} b_j^2 H(w_j, w_j)$$
$$= \sum_{j=1}^{r} b_j^2 H(w_j, w_j)$$
$$= \sum_{j=1}^{q} b_j^2 H(w_j, w_j) \geq 0$$

So, $H(v_0, v_0) < 0$ and $H(v_0, v_0) \geq 0$, a contradiction. So, we conclude that $p = q$.

Definition 4.10.2 The number of positive diagonal entries in a ~~matrix~~ diagonal representation of a symmetric bilinear form on a real vector space is called the index of the form. The difference between the number of positive and the number of negative diagonal entries in a diagonal representation of a symmetric bilinear form is called the signature of the form. The three terms rank, index and signature are called the invariants of the bilinear form because they are invariants with respect to matrix representations. These same terms apply to the associated quadratic form. Notice that the values of any two of these invariants determine the value of the third.

Example 4 The bilinear form corresponding to the quadratic form K of Example 3 (Page-74) has a 3×3 diagonal matrix representation with diagonal entries of 2, 7 and 0. So, the rank, index, and signature are each 2.