Example 5 The matrix representation of the bilinear form corresponding to the quadratic form $K(x,y) = x^2 - y^2$ on $\mathbb{R}^2$ with respect to the standard ordered basis is the diagonal matrix with diagonal entries 1 and -1. So, the rank of K is 2, the index of K is 1 and the signature of K is 0.

Since the congruence relation is intimately associated with bilinear forms, we can apply Sylvester's Law of inertia to study this relation on the set of real symmetric matrices. Let A be a real $n \times n$ symmetric matrix and suppose that D and E are each diagonal matrices congruent to A. By Corollary 4.4.8 of Theorem 4.4.5, A is the matrix representation of the bilinear form H on $\mathbb{R}^n$ defined by $H(x,y) = x^t A y$ with respect to the standard ordered basis for $\mathbb{R}^n$. So, Sylvester's law of inertia tells us that D and E have the same number of positive and negative diagonal entries. We can state this result as the matrix version of Sylvester's Law.

Corollary 4.10.2 Let A be a real symmetric matrix. Then the number of positive diagonal entries and the number of negative diagonal entries in any diagonal matrix congruent to A is independent of the choice of the diagonal matrix.

Definition 4.10.3 Let A be a real symmetric matrix, and let D be a diagonal matrix that is congruent to A. The number of positive diagonal entries of D is called the index of A. The difference between the number of positive diagonal entries and the ~~neg~~ number of negative diagonal entries of D is called the signature of A. As before, the rank, index and signature of a matrix are called the invariants of the matrix and the values of any two of these invariants determine the value of the third.

Corollary 4.10.4 Two real symmetric $n\times n$ matrices are congruent if and only if they have the same invariants.

Proof: If A and B are $n\times n$ congruent symmetric matrices, then they are both congruent to the same diagonal matrix, and it follows that they have the same invariants.

Conversely, suppose that A and B are $n\times n$ symmetric matrices, with the same invariants. Let D and E ~~are~~ be the diagonal matrices congruent to A and B respectively, chosen so that the diagonal entries are in the order of positive, negative and zero. Since A and B have the same invariants, so do D and E. Let p and r denote the index and rank ~~of~~ respectively of both D and E. Let d_i denotes the ith diagonal entry of D, let Q be the $n\times n$ diagonal matrix whose ith diagonal entry q_i is given by

$$q_i = \begin{cases} \dfrac{1}{\sqrt{d_i}} & \text{if } 1 \le i \le p \\ \dfrac{1}{\sqrt{-d_i}} & \text{if } p < i \le r \\ 1 & \text{if } r < i \end{cases}$$

Then $Q^t D Q = J_{pr}$ where

$$J_{pr} = \begin{pmatrix} I_p & O & O \\ O & -I_{r-p} & O \\ O & O & O \end{pmatrix}$$

It follows that A is congruent to J_{pr}. Similarly B is congruent to J_{pr}, and hence A is congruent to B.

The matrix J_{pr} acts as a canonical form for the theory of real symmetric matrices. The next corollary whose proof is contained in ~~corollary 4.10.4~~ the proof of Corollary 4.10.4, describes the role of J_{pr}

Corollary 4.10.5 A real symmetric ~~matrices~~ matrix $n\times n$ matrix A has index p and rank r if and only if A is congruent to J_{pr} (as just defined)

Example 6

Let $A = \begin{pmatrix} 1 & 1 & -3 \\ -1 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

We apply corollary ~~4.10.4~~ 4.10.4 to determine which pair of the matrices A, B and C are congruent.

The matrix A is the 3×3 matrix in Example 1 (Page-67), where it is shown that A is congruent to a diagonal matrix with diagonal entries 1, 1 and -24. Therefore, A has rank 3 and index 2. Also it can be shown that B and C are congruent, respectively, to the diagonal matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

It follows that both A and C have rank 3 and index 2, while B has rank 3 and index 1. We conclude that A and C are congruent but that B is ~~neither~~ congruent ~~neither~~ to neither A nor C.

4.11 Dual spaces

In this section, we are concerned exclusively with linear transformations from a vector space V into its field of scalars F, which is itself a vector space of dimension 1 over F. Such a linear transformation is called a linear functional on V. We generally use the letters $f, g, h, \ldots$ to denote linear functionals. As we see in Example 1, the definite integral provides us with one of the most important examples of a linear functional in mathematics.

Example 1 Let V be the vector space of continuous real valued functions on the interval $[0, 2\pi]$. Fix a function $g \in V$. The function $h : V \to \mathbb{R}$ defined by

$$h(x) = \frac{1}{2\pi}\int_0 x(t)g(t)\,dt \qquad \text{is a}$$

linear functional on V. In the cases, $g(t)$ equals $\sin nt$ or $\cos nt$, $h(x)$ is often called the nth Fourier coefficient of x.

Example 2 Let $V = M_{n\times n}(F)$, and define $f : V \to F$ by $f(A) = tr(A)$, where $tr(A)$ denotes the trace of A. As $tr(A+B) = tr(A) + tr(B)$ and $tr(cA) = c\,tr(A)$ for $c \in F$, and $A, B \in V$. So f is a linear functional.

Example 3. Let V be a finite dimensional vector space, and