Let $\beta = \{x_1, x_2, \dots, x_n\}$ be an ordered basis for V. For each $i = 1, 2, \dots, n$, define $f_i(x) = a_i$ where

$$[x]_\beta = \text{coordinates vector of } x \text{ with respect to the ordered basis } \beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

for $x \in V$. Then f_i is a linear functional on V, called the ith coordinate function with respect to the basis β. Note that $f_i(x_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. These linear functionals play an important role in the theory of dual spaces (see Theorem 4.11.2)

Definition 4.11.1 For a vector space V over F, we define the dual space of V to be the vector space $L(V, F)$ of linear functionals on V, denoted by V^*.

[Note: We know that, if V and W are two vector spaces over a field F, then $L(V, W)$ = the set of all linear transformation from V to W forms a vector space over F with respect to the vector addition $T+S$ and scalar multiplication cT defined by $(T+S)(x) = T(x) + S(x)$ and $(cT)(x) = c(T(x))$ for $T, S \in L(V, W)$ and $c \in F$.]

So, V^* is the vector space consisting all linear functionals on V with the operations of addition and scalar multiplication defined as above. Note that if V is finite dimensional, then from our previous result on linear transformation,

$$\dim(V^*) = \dim\left(L(V, F)\right) = \dim(V) \cdot \dim(F) = \dim V.$$

Hence from our early result V and V^* are isomorphic. We also define the double dual V^{**} of V to be dual of V^*. In Theorem , we show, in fact, that there is a natural identification of V and V^{**} in the case that V is finite dimensional.

Theorem 4.11.2 Suppose that V is a finite dimensional vector space with the ordered basis $\beta = \{x_1, x_2, \ldots, x_n\}$. Let f_i $(1 \le i \le n)$ be the ith coordinate function with respect to β as just defined on Page-89, and let $\beta^* = \{f_1, f_2, \ldots, f_n\}$. Then β^* is an ordered basis for V^*, and, for any $f \in V^*$, we have $f = \sum_{i=1}^{n} f(x_i) f_i$

Proof: Let $f \in V^*$. As $\dim(V^*) = n$, we need only show that $f = \sum_{i=1}^{n} f(x_i) f_i$ from which it follows that β^* generates V^* and hence a basis of V^* as $\dim(V^*) = n$ (from ~~p~~ our previous result on vector space).

Let $g = \sum_{i=1}^{n} f(x_i) f_i$,

For $1 \le j \le n$ $\quad g(x_j) = \left(\sum_{i=1}^{n} f(x_i) f_i\right)(x_j)$

$$= \sum_{i=1}^{n} f(x_i) f_i(x_j) = \sum_{i=1}^{n} f(x_i)\, \delta_{ij} = f(x_j)$$

as $\delta_{ij} = 1$ for $i = j$
$\quad\;\; = 0$ for $i \ne j$.

We know that if V and W are finite dimensional vector space and ~~T: V → W be a linear~~ T and S are two linear transformation from V to W and if $T(v_i) = S(v_i)$ for some ordered basis $\{v_1, v_2, \ldots, v_n\}$ of V (whose dimension is n) then $T = S$.

So, using this result, $\quad g = f = \sum_{i=1}^{n} f(x_i) f_i$.

Definition 4.11.3 Using notation of Theorem 4.11.2, we call the ordered basis $\beta^* = \{f_1, f_2, \ldots, f_n\}$ of V^* that satisfies

$f_i(x_j) = \delta_{ij}$ $(1 \le i, j \le n)$ the dual basis of β.

Example 4 Let $\beta = \{(2,1),(3,1)\}$ be an ordered basis for $\mathbb{R}^2$ suppose that the dual basis for β is given by $\beta^* = \{f_1, f_2\}$ To explicitly determine a formula for f_1, we need to consider the equations

$$1 = f_1(2,1) = f_1(2e_1 + e_2) = 2f_1(e_1) + f_1(e_2), \quad e_1 = (1,0),\ e_2 = (0,1)$$
$$0 = f_1(3,1) = f_1(3e_1 + e_2) = 3f_1(e_1) + f_1(e_2)$$

Solving these equations $f_1(e_1) = -1$ and $f_1(e_2) = 3$, that is, $f_1(x,y) = -x + 3y$. Similarly it can be shown that $f_2(x,y) = x - 2y$.

We now assume that V and W are finite dimensional vector spaces over F with ordered basis β and γ respectively. We have proved earlier that there is a one-to-one correspondence between linear transformations $T: V \to W$ and $m \times n$ matrices (over F) via the correspondence $T \leftrightarrow [T]_\beta^\gamma$ where $[T]_\beta^\gamma$ is the matrix of T relative to the order basis β of V and ordered basis γ of W. For a matrix of the form $A = [T]_\beta^\gamma$, the question arises as to whether or not there exists a linear transformation U associated with T in some natural way such that U may be represented in some basis as A^t. Of course, if $m \neq n$, it would be impossible for U to be a linear transformation from V to W. We now answer this question by applying what we have already learned about dual spaces.

Theorem 4.11.4 Let V and W be finite dimensional vector spaces over F with ordered bases β and γ, respectively. For any linear transformation $T: V \to W$, the mapping $T^t: W^* \to V^*$ defined by $T^t(g) = gT$ for all $g \in W^*$, is a linear transformation with the property that $[T^t]_{\gamma^*}^{\beta^*} = \left([T]_\beta^\gamma\right)^t$

Proof: For $g \in W^*$, it is clear that $T^t(g) = gT$ is a linear functional on V and hence is in V^*. So, T^t maps W^* into V^*. We can easily verify that T^t is linear (verify it).

To complete the proof, let $\beta = \{x_1, x_2, \ldots, x_n\}$ and $\gamma = \{y_1, y_2, \ldots, y_n\}$ with dual basis $\beta^* = \{f_1, f_2, \ldots, f_n\}$ and $\gamma^* = \{g_1, g_2, \ldots, g_m\}$ respectively.

For Convenience, let $A = [T]_\beta^\gamma = [a_{ij}]_{m\times n}$. To find the jth column of $[T^t]_{\gamma^*}^{\beta^*}$, we begin by expressing $T^t(g_j)$ as a linear combination of the vectors for β^*.

By Theorem 4.11.2, we have

$$T^t(g_j) = g_j T = \sum_{s=1}^{n} (g_j T)(x_s) f_s .$$

So, the row i, column j entry of $[T^t]_{\gamma^*}^{\beta^*}$ is

$$(g_j T)(x_i) = g_j(T(x_i)) = g_j\left(\sum_{k=1}^{m} a_{ki} y_k\right)$$

$$= \sum_{k=1}^{m} a_{ki}\, g_j(y_k) = \sum_{k=1}^{m} a_{ki}\, \delta_{jk} = a_{ji}$$

Hence $[T^t]_{\gamma^*}^{\beta^*} = A^t$.
