The linear transformation T^t defined in Theorem 4.11.4 is called the transpose of T. It is clear that T^t is unique linear transformation U such that $[U]_{\gamma^*}^{\beta^*} = \left([T]_\beta^\gamma\right)^t$.

We illustrate Theorem 4.11.4 with the next example.

Example 5 Define $T : P_1(\mathbb{R}) \to \mathbb{R}^2$ by $T(p(x)) = (p(0), p(2))$

Let β and γ be the standard ordered basis for $P_1(\mathbb{R})$ and $\mathbb{R}^2$, respectively. Clearly $[T]_\beta^\gamma = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

We compute $[T^t]_{\beta^*}^{\gamma^*}$ directly from the definition.

Let $\beta^* = \{f_1, f_2\}$ & $\gamma^* = \{g_1, g_2\}$

Suppose that $[T^t]_{\gamma^*}^{\beta^*} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $T^t(g_1) = af_1 + cf_2$. So,

$$T^t(g_1)(1) = (af_1 + cf_2)(1) = a f_1(1) + c f_2(1) = a(1) + c(0) = a$$

But also $T^t(g_1)(1) = g_1(T(1)) = g_1(1,1) = 1$

So, $a = 1$. Using similar computations, we obtain that $c = 0$, $b = 1$ and $d = 2$. Hence a direct computation gives $[T^t]_{\gamma^*}^{\beta^*} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \left([T]_\beta^\gamma\right)^t$ as predicted in Theorem 4.11.4.

We now concern ourselve with demonstrating that any finite dimensional vector space V can be identified in a natural way with its double dual V^{**}. There is, in fact, an isomorphism V and V^{**} that does not depend on any choice of bases of for the the vector spaces.

For a vector $x \in V$, we define $\hat{x} : V^* \to F$ by

$\hat{x}(f) = f(x)$ for every $f \in V^*$. It is to verify that $\hat{x}$ is a linear functional on V^* (verify it). So, $\hat{x} \in V^{**}$

The correspondence $x \longleftrightarrow \hat{x}$ allows us the desired isomorphism between V and V^{**}.

Lemma 4.11.5 Let V be a finite dimensional vector space, and let $x \in V$. If $\hat{x}(f) = 0$ for all $f \in V^*$, then $x = \theta$.

Proof: Let $x \neq \theta$. We show that there exists $f \in V^*$ such that $\hat{x}(f) \neq 0$. Choose an ordered basis $\beta = \{x_1, x_2, \ldots, x_n\}$ for V such that $x_1 = x$. Let $\{f_1, f_2, \ldots, f_n\}$ be the dual basis of β. Then $f_1(x_1) = 1 \neq 0$. Let $f = f_1$.

Theorem 4.11.6 Let V be a finite dimensional vector space and define $\psi : V \longrightarrow V^{**}$ by $\psi(x) = \hat{x}$. Then ψ is an isomorphism.

Proof: First we show that ψ is linear. Let $x, y \in V$. For $f \in V^*$, we have

$$\psi(cx+y)(f) = f(cx+y) = cf(x) + f(y)$$
$$= c\,\hat{x}(f) + \hat{y}(f) = (c\hat{x} + \hat{y})(f)$$

So, $\psi(cx+y) = c\hat{x} + \hat{y} = c\psi(x) + \psi(y)$ for $c \in F$.

Now we show that ψ is injective. Suppose the $\psi(x)$ is the zero functional on V^* for some $x \in V$. Then $\hat{x}(f) = 0$ for every $f \in V^*$. By the Lemma 4.11.5, we conclude that $x = \theta$.

So, ψ is an isomorphism as ψ is linear, injective and $\dim(V) = \dim(V^{**})$.

Corollary 4.11.7 Let V be a finite dimensional vector space with dual space V^*. Then every ordered basis for V^* is the dual basis for some basis of ~~Vo~~ for V.

Proof: Let $\{f_1, f_2, \ldots, f_n\}$ be an ordered basis for V^*. We may combine Theorems 4.11.2 and Theorem 4.11.6 to conclude that for this basis for V^* there exists a dual basis $\{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n\}$ in V^{**}, that is $\delta_{ij} = \hat{x}_i(f_j) = f_j(x_i)$ for all i and j. Thus $\{f_1, f_2, \ldots, f_n\}$ is the dual basis of $\{x_1, x_2, \ldots, x_n\}$.

Although many of the ideas of this section (e.g., the existence of a dual space), can be extended to the case where V is not finite dimensional, only a finite dimensional vector space is ~~isomorphod~~ isomorphic to its double dual via the map $x \to \hat{x}$. In fact, for infinite dimensional vector spaces, no two V, V^*, and V^{**} are isomorphic.

Definition 4.11.8 (Annihilator) Let V be a finite dimensional vector space over a field F. For every subset S of V, define the annihilator S^0 of S as

$$S^0 = \{f \in V^* : f(x) = 0 \text{ for all } x \in S\}$$

Theorem 4.11.9 Let V be a finite dimensional vector space over a field F. Then

(a) If S be a subset of S then S^0 is a subspace of V^*

~~(b) $(S^0)^0 = \operatorname{span}(\psi(S))$, where ψ is defined as in Theorem 4.11.6~~

(c) If W is a subspace of V and $x \notin W$ Then there exists

$f \in W^0$ such that $f(x) \neq 0$

(c) $(S^0)^0 = \mathrm{span}(\psi(S))$ where ψ is defined as in Theorem 4.11.6

(d) For subspaces W_1 and W_2, prove that $W_1 = W_2$ if and only if $W_1^0 = W_2^0$

(e) For subspaces W_1, W_2, show that $(W_1+W_2)^0 = W_1^0 \cap W_2^0$

Proof: (a) The zero function is an element of S^0, S^0 is non-empty. Let $f, g \in S^0$. So, $f, g \in V^*$ and $f(x) = 0$, $g(x) = 0$ for all $x \in S$. So, $f+g \in V^*$ and $(f+g)(x) = f(x) + g(x) = 0+0 = 0$ for all $x \in S$. So, $f+g \in S^0$ and $cf \in V^*$ for $c \in F$, $f \in S^0$ and $(cf)(x) = c(f(x)) = c.0 = 0$ for all $x \in S$. So, $cf \in S^0$. So, S^0 is a subspace of V^*

(b) As V is finite dimensional, W is finite dimensional. Let $\{v_1, v_2, \ldots, v_k\}$ be a basis of W. Since $x \notin W$, so, $\{v_1, v_2, \ldots, v_k, v_{k+1}\}$ is a linearly independent set where $v_{k+1} = x$ and $\{v_1, v_2, \ldots v_{k+1}\}$ can be extended to a basis $\{v_1, v_2, \ldots, v_n\}$ for V. So, we can define a linear functional f on V such that $f(v_i) = \delta_{i(k+1)}$. Thus f is the desired functional.

(c) Let $W = \mathrm{span}(S)$. We first prove that $W^0 = S^0$. As $S \subset \mathrm{span}(S) = W$. So, if $f \in V^*$ and $f(x) = 0$ for all $x \in W$ then $f(x) = 0$ for all $x \in S$. So, $W^0 \subset S^0$. Now let $f \in S^0$. So,