When testing T for diagonalizability, it is usually ~~easist~~ easiest to choose a convenient basis α for V and work with $B = [T]_{\alpha}$. If the characteristic polynomial of B splits, then use the condition 2 above to check if the multiplicity of each repeated eigenvalue of B equals $n - \text{rank}(B - \lambda I)$. (By Theorem 4.12.2, condition 2 is automatically satisfied for eigenvalues of multiplicity 1.) If so, then B, and hence T, is diagonalizable.

If T is diagonalizable and a basis β for V consisting of eigenvectors of T is desired, then we first find a basis for each eigenspace of B. The union of these bases is a basis γ for F^n consisting of eigenvectors of B. Each vector in γ is the coordinate vector relative to α of an eigenvector of T. The set consisting of these n eigenvectors of T is the desired basis β.

Furthermore, if A is an $n \times n$ diagonalizable matrix. Then we know that we can find an invertible $n \times n$ matrix Q a diagonal $n \times n$ matrix D such that $Q^{-1} A Q = D$. The matrix Q has its columns the vectors in a basis of eigenvectors of A, and D has as its jth diagonal entry the eigenvalue of A corresponding to the jth column of Q.

We now consider some examples to illustrate the preceding ideas.

Example 3 We test the matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$$

for diagonalizability. The characteristic polynomial of A is $\det(A - tI) = -(t-4)(t-3)^2$, which splits, and so condition 1 of the test of diagonalization is satisfied. Also A has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 3$ with multiplicities 1 and 2 respectively. Since λ_1 has multiplicity 1, condition 2 is satisfied for λ_1. Thus we need only test condition 2 for λ_2.

Because $A - \lambda_2 I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has rank 2,

we see that $3 - \operatorname{rank}(A - \lambda I) = 1$ which is not the multiplicity of λ_2. Thus condition 2 fails for λ_2, and so, A is not diagonalizable.

Example 4 Let T be the linear operator on $P_2(\mathbb{R})$ defined by

$$T(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2.$$

We first test T for diagonalizability. Let α denote the standard ordered basis for $P_2(\mathbb{R})$ and $B = [T]_\alpha$. Then $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

The characteristic polynomial of B, and hence of T, is $-(t-1)^2(t-2)$, which splits. Hence condition 1 is satisfied for test of diagonalizability. Also B has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ with multiplicities 2 and 1, respectively. Condition 2 is satisfied for λ_2 because it has multiplicity 1. So we need only verify condition 2 for $\lambda_1 = 1$.

For this case,

$$3 - \text{rank}(B - \lambda_1 I) = 3 - \text{rank}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 3 - 1 = 2$$

So, the multiplicity of $\lambda = 3 - \text{rank}(B - \lambda_1 I)$. So, T is diagonalizable.

We now find an ordered basis γ for $\mathbb{R}^3$ of eigenvectors of B. We consider each eigenvalue separately.

The eigenspace corresponding $\lambda_1 = 1$ is

$$E_{\lambda_1} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Which is the solution space of the system

$$x_2 + x_3 = 0$$

and has $\gamma_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$ as a basis. ~~for E_{λ_1}~~.

The eigenspace corresponding to $\lambda_2 = 2$ is

$$E_{\lambda_2} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Which is the solution space of the system

$$-x_1 + x_2 + x_3 = 0$$
$$x_2 = 0$$

and has $\gamma_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ as a basis.

Let $\gamma = \gamma_1 \cup \gamma_3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Then γ is an ordered basis for $\mathbb{R}^3$ consisting of eigenvectors of B.

Finally, observe that the vectors in γ are the coordinate vectors relative to α of the vectors in the set $\beta = \{1, -x+x^2, 1+x^2\}$ which is an ordered basis for $P_2(\mathbb{R})$ consisting of eigenvectors of T. ~~Thus $[T]_\beta =$~~

So, $[T]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Our next example is an application of diagonalization.

<u>Example 5</u> Let $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$

We show that A is diagonalizable and find a 2×2 matrix Q such that $Q^{-1}AQ$ is a diagonal matrix. We then show how to use this result to compute A^n for any positive integer n.

First observe that the characteristic polynomial of A is $(t-1)(t-2)$, and hence A has two distinct eigenvalues, $\lambda_1 = 1$ and $\lambda_2 = 2$. By applying Corollary 4.13.3 of Theorem 4.13.2 to the operator L_A, we see that A is diagonalizable. Moreover, $\gamma_1 = \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$ and $\gamma_2 = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ are bases for