$f(x) = 0$ for all $\in S$. Let $y \in W = span(S)$. So,

$y = \sum_{i=1}^{m} c_i x_i$ $c_1, c_2, \ldots c_m \in F$ and $x_1, x_2, \ldots, x_m \in S$. So,

$f(y) = f\left(\sum_{i=1}^{n} c_i x_i\right) = \sum_{i=1}^{m} c_i f(x_i) = 0$. So, $f \in W^0$

So, $S^0 \subset W^0$. Hence $W^0 = S^0$

Since $(W^0)^0 = (S^0)^0$ and $span(\psi(S)) = \psi(W)$ as ψ is an isomorphism, so we just prove that $(W^0)^0 = \psi(W)$

Now, by Theorem 4.11.6 we may assume that every element in $(W^0)^0 \subset V^{**}$ has the form $\hat{x}$ for some $x \in V$.

Let $\hat{x}$ be an element of $(W^0)^0$. So, we have $\hat{x}(f) = f(x) = 0$ if $f \in W^0$. Now if x is not an element ~~of~~ in W, by the previous result (b), $\exists$ some functional $f \in W^0$ such that $f(x) \neq 0$. But this is a contradiction. So, $x \in W \Rightarrow \psi(x) = \hat{x} \in \psi(W)$

So, $(W^0)^0 \subset \psi(W)$

For the converse, we may assume that $\hat{x}$ is an element in $\psi(W)$. Thus for all $f \in W^0$, we have $\hat{x}(f) = f(x) = 0$ since x is an element of W. So $\hat{x} \in (W^0)^0$. So, $\psi(W) \subset (W^0)^0$

Hence $(W^0)^0 = \psi(W)$ and the result is proved.

(d) If $W_1 = W_2$ then by definition $W_1^0 = W_2^0$

Conversely, let $W_1^0 = W_2^0$ then we have

$\psi(W_1) = (W_1^0)^0 = (W_2^0)^0 = \psi(W_2)$ from (c) and hence $W_1 = W_2$ as ψ is an isomorphism.

(e) Let $f \in (W_1+W_2)^0$. Then $f(w_1+w_2)=0$ for all $w_1 \in W_1$ and for all $w_2 \in W_2$. Let ~~w~~ $x_1 \in W_1$. Then $x_1 = x_1 + \theta \in W_1+W_2$, so $f(x_1)=0$. Let $x_2 \in W_2$. Then ~~$x_2 = x_2 + \theta$~~ ~~W_1+W_2~~ $x_2 = \theta + x_2 \in W_1+W_2$. So, $f(x_2)=0$. Hence $f(x)=0$ for all $x \in W_1$. So $f \in W_1^0$. Also $f(x)=0$ for all $x \in W_2$. So, $f \in W_2^0$. Hence $f \in W_1^0 \cap W_2^0$

Hence $(W_1+W_2)^0 \subset W_1^0 \cap W_2^0$

Conversely, let $f \in W_1^0 \cap W_2^0$. Let $w \in W_1+W_2$

Then $w = w_1+w_2$, $w_1 \in W_1$, $w_2 \in W_2$. As $f \in W_1^0 \cap W_2^0$

$f(w_1)=0$, $f(w_2)=0$

Now $f(w) = f(w_1+w_2) = f(w_1)+f(w_2) = 0+0 = 0$

So, $f(w)=0$ for all $w \in W_1+W_2$

So, $f \in (W_1+W_2)^0$. Hence $W_1^0 \cap W_2^0 \subset (W_1+W_2)^0$

Hence $(W_1+W_2)^0 = W_1^0 \cap W_2^0$

Theorem: 4.11.10 Let V be a finite dimensional vector space over a field F. If W be a subspace of V, then $\dim(W) + \dim(W^0) = \dim V$

Proof: Let $\{x_1, x_2, \ldots, x_k\}$ be an ordered basis of W. So, $\{x_1, x_2, \ldots, x_k\}$ can be extended to an ordered basis $\beta = \{x_1, x_2, \dots x_n\}$ of V. ~~Let β^* be~~ let $\beta^* = \{f_1, f_2, \dots, f_n\}$ be the dual basis ~~of~~ of β in V^*. We ~~can~~ are going to prove that $\alpha = \{f_{k+1}, f_{k+2}, \dots, f_n\}$ is a basis for W^0. For this, we only need to prove that ~~α~~ $span(\alpha) = W^0$ since as $\alpha \subset \beta^*$, we already know that α is linearly independent set. Since $W^0 \subset V^*$, every element $f \in W^0$ can be written as $f = \sum_{j=1}^{n} a_j f_j$, $a_j \in F$, $i = 1, 2, \dots, n$

Now $x_i \in W$ for $i = 1, 2, \dots, k$

So, $0 = f(x_i) = \left(\sum_{j=1}^{n} a_j f_j\right)(x_i) = \sum_{j=1}^{n} a_i f_j(x_i) = a_i$, $i = 1, 2, \dots, k$

So $f = \sum_{j=k+1}^{n} a_j f_j$ So, $f \in span(\alpha)$

So, $\dim(W^0) = n - k$ Hence $\dim(W) + \dim(W^0) = n = \dim V$.

Theorem 4.11.11 Let V and W are two finite dimensional vector spaces over a field F, and $T: V \to W$ is linear. Then $N(T^t) = (R(T))^0$ where $N(T^t)$ is the null space of T^t and $R(T)$ is the range space of T.

Proof: Let $f \in N(T^t)$, so $T^t(f) = fT = 0$ --(1)

Let $y \in R(T)$ ~~. So~~ y ~~$= T(x)$~~ ~~for~~ some $x \in V$, Then from (1), $f(y) = 0$ for all $y \in R(T)$ and hence $f \in (R(T))^0$

Now, if $f \in (R(T))^0$, this means $f(y) = 0$ for all $y \in R(T)$ and hence $T^t(f(x)) = f(T(x)) = 0$ for all x. This means $f \in N(T^t)$. So, $(R(T))^0 = N(T^t)$.

4.12 Eigenspaces of a linear operator

Definition 4.12.1 (Eigenspace of a Linear operator):
Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. Define $E_\lambda = \{x \in V : T(x) = \lambda x\}$ $= N(T - \lambda I_V)$ = Null space of $T - \lambda I_V$, I_V is the identity operator on V.

The set E_λ is called the eigenspace of T corresponding to the eigenvalue λ. Analogously, we define the eigenspace of a square matrix A to be the eigenspace of the linear operator $L_A : F^n \to F^n$ defined by $L_A(x) = Ax$.

Clearly, E_λ is a subspace of V corresponding to the zero vector and the eigenvectors of T corresponding to the eigenvalue λ (Prove it). The maximum number of linearly independent eigenvectors of T corresponding to the eigenvalue λ is the dimension of E_λ. Our next result relates this dimension to the multiplicity of λ.

Theorem 4.12.2 Let T be a linear operator on a finite dimensional vector space, and let λ be an eigenvalue of T having multiplicity m. Then $1 \le \dim(E_\lambda) \le m$

Proof: Choose an ordered basis $\{v_1, v_2, \ldots, v_p\}$ of E_λ