Extend it to an ordered basis $\beta = \{v_1, v_2, \ldots v_p, v_{p+1}, \ldots, v_n\}$ for V and let $A = [T]_\beta$. Observe that v_i, ~~i=1,2,...,p~~ is an eigen vector of T corresponding to λ, $i=1,2,\ldots,p$.

So,
$$A = \begin{pmatrix} \lambda I_p & B \\ O & C \end{pmatrix}$$

So, the Characteristic polynomial of T is

$$f(t) = \det(A - \lambda I_n) = \det \begin{pmatrix} (\lambda - t) I_p & B \\ O & C - tI_{n-p} \end{pmatrix}$$

$$= \det((\lambda - t) I_p) \det(C - tI_{n-p})$$

$$= (\lambda - t)^p g(t), \text{ where } g(t) \text{ is a polynomial.}$$

Thus $(\lambda - t)^p$ is a factor of $f(t)$ and hence the multiplicity of λ is atleast p. But $\dim(E_\lambda) = p$ and so $\dim(E_\lambda) \leq m$. As clearly $\dim(E_\lambda) \geq 1$

So, $1 \leq \dim(E_\lambda) \leq m$

4.13 Diagonalizability

In our previous linear algebra course, we presented the diagonalization problem and observed that not all linear operators or matrices are diagonalizable. We state a necessary and sufficient condition of diagonalizability which we have proved earlier

Theorem 4.13.1 A linear operator T on a finite dimensional vector space V is diagonalizable if and only if $\exists$ an ordered basis β for V consisting of eigenvectors of T. Furthermore, if T is diagonalizable, $\beta = \{v_1, v_2, \ldots, v_n\}$ is an ordered basis of eigenvectors of T, $D = [T]_\beta = [d_{ij}]_{n\times n}$. Then D is a diagonal matrix and d_{jj} is the eigenvalue corresponding to v_j, $j = 1, 2, \ldots, n$.

We have ~~yet~~ not ~~solve~~ yet solved the diagonalization problem. What is still needed is a simple test to determine whether an operator or a matrix can be diagonalized, as well as a method for actually finding a basis of eigenvectors. Here we develop such a test and method.

Theorem 4.13.2 Let T be a linear operator on a vector space V, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T. If $v_1, v_2, \ldots, v_k$ are eigenvectors of T corresponding to λ_i, eigenvector is v_i, $i = 1, 2, \ldots, k$, then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Proof: The proof is by mathematical induction on k. Suppose that $k=1$, Then $v_1 \neq \theta$ since v_1 is an eigenvector, hence $\{v_1\}$ is linearly independent. Now assume the theorem holds for $k-1$ distinct ~~eigenvectors~~ eigenvalues, where $k-1 \geq 1$, and that we have ~~vectors~~ k eigenvectors $v_1, v_2, \ldots, v_k$ corresponding to the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. We wish to show that $\{v_1, v_2, \ldots, v_k\}$ is linearly independent. Suppose that

$a_1, a_2, \dots, a_k$ are scalars such that $\sum_{i=1}^{k} a_i v_i = \theta \quad \cdots (1)$

Applying $(T - \lambda_k I)$ to both sides of (1), we obtain

$$a_1(\lambda_1 - \lambda_k)v_1 + a_2(\lambda_2 - \lambda_k)v_2 + \dots + a_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1} = \theta$$

By the induction hypothesis $\{v_1, v_2, \dots, v_{k-1}\}$ is linearly independent, and hence

$$a_1(\lambda_1 - \lambda_k) = a_2(\lambda_2 - \lambda_k) = \dots = a_{k-1}(\lambda_{k-1} - \lambda_k) = 0$$

Since, $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct, it follows that $\lambda_i - \lambda_k \neq 0$ for $i = 1, 2, \dots, k-1$. So, $a_1 = a_2 = \dots = a_{k-1} = 0$ but $v_k \neq 0$ and therefore $a_k = 0$. Consequently $a_1 = a_2 = \dots = a_k$, and it follows that $\{v_1, v_2, \dots, v_k\}$ is linearly independent.

Corollary 4.13.3 Let T be a linear operator on a n dimensional vector space V. If T has n distinct eigenvalues, then T is diagonalizable.

Proof: Suppose that T has n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. For each i, choose an eigenvector v_i corresponding to λ_i. By Theorem 4.13.2, $\{v_1, v_2, \dots, v_n\}$ is linearly independent and since $\dim(V) = n$, this set is a basis for V. So, by Theorem 4.13.1, T is diagonalizable.

Example 1 Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$.

The characteristic polynomial of A (and hence of L_A) is

$$\det(A - tI) = \det\begin{pmatrix} 1-t & 1 \\ 1 & 1-t \end{pmatrix} = t(t-2).$$

So, the eigenvalues of L_A are 0 and 2. Since L_A is a linear operator on two dimensional vector space $\mathbb{R}^2$, we conclude from the preceding corollary that L_A (and hence A) is diagonalizable.

The converse of corollary 4.13.3 of Theorem 4.13.2 is false. That is, it is not true that if T is diagonalizable, then it has n distinct eigenvalues. For example, the identity operator is diagonalizable even though it has only one eigenvalue, namely, $\lambda = 1$.

We have seen that diagonalizability requires the existence of eigenvalues. Actually, ~~diag~~ diagonalizability imposes a stronger condition on the characteristic polynomial.

Definition 4.13.4 A polynomial $f(t)$ in $P(F)$ splits over F if there are scalars, $c, a_1, a_2, \ldots, a_n$ (not necessarily distinct) in F such that

$$f(t) = c(t-a_1)(t-a_2)\cdots(t-a_n),$$

For example, $t^2-1 = (t+1)(t-1)$ splits over $\mathbb{R}$, but $(t^2+1)(t-2)$ does not split over $\mathbb{R}$ because t^2+1 can not be factored into a product of linear factors over $\mathbb{R}$.