However $(t^2+1)(t-2)$ does split over $\mathbb{C}$ because it factors into the product $(t+i)(t-i)(t-2)$. If $f(t)$ is the characteristic polynomial of a linear operator or a matrix over a field F, then the statement that $f(t)$ splits is understood to mean that it splits over F.

Theorem 4.13.5 The characteristic polynomial of any diagonalizable linear operator splits.

Proof: Let T be a diagonalizable linear operator on the n dimensional vector space V, and let β be an ordered basis for V such that $[T]_\beta = D$ is a diagonal matrix. Suppose that

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

and let $f(t)$ be the characteristic polynomial of T.

Then $f(t) = \det(D - tI) = \begin{vmatrix} \lambda_1 - t & 0 & \cdots & 0 \\ 0 & \lambda_2 - t & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \lambda_n - t \end{vmatrix}$

$$= (\lambda_1 - t)(\lambda_2 - t)\cdots(\lambda_n - t) = (-1)^n (t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_n)$$

So, $f(t)$ splits.

From this theorem, it is clear that if T is a diagonalizable linear operator on an n dimensional vector space that fails to have distinct eigenvalues, then the characteristic polynomial of T must have repeated zeros. The Converse of the Theorem 4.13.5 is false, that is, the characteristic polynomial of T may split but T need not be diagonalizable. Consider the following example: Let T be a linear operator on $P_2(\mathbb{R})$ defined by $T(f(x)) = f'(x)$. The matrix representation of T with respect to the standard ordered basis β for $P_2(\mathbb{R})$ is

$$[T]_\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Consequently, the characteristic polynomial of T is

$$\det\left([T_\beta] - tI\right) = \left|\begin{bmatrix} -t & 1 & 0 \\ 0 & -t & 2 \\ 0 & 0 & -t \end{bmatrix}\right| = -t^3$$

Thus T has only one eigenvalue $(\lambda = 0)$ with multiplicity 3. Solving $T(f(x)) = f'(x) = 0$ shows that $E_\lambda = N(T - \lambda I) = N(T)$ is a subspace of $P_2(\mathbb{R})$ consisting of the constant polynomials. So $\{1\}$ is a basis for E_λ and so, $\dim(E_\lambda) = 1$. Consequently, there is ~~not~~ no basis for $P_2(\mathbb{R})$ consisting of eigenvectors of T and so T is not diagonalizable.

<u>Another Example</u> Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 4a_1 + a_3 \\ 2a_1 + 3a_2 + 2a_3 \\ a_1 + 4a_3 \end{pmatrix}$$

we determine the eigenspace of T corresponding to each eigenvalue. Let β be the standard ordered basis for $\mathbb{R}^3$.

Then $[T]_\beta = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$

and hence the characteristic polynomial of T is

$$\det\left([T]_\beta - tI\right) = \begin{vmatrix} 4-t & 0 & 1 \\ 2 & 3-t & 2 \\ 1 & 0 & 4-t \end{vmatrix} = -(t-5)(t-3)^2$$

So, the eigenvalues of T are $\lambda_1 = 5$ and $\lambda_2 = 3$ with multiplicities 1 and 2 respectively

Since $E_{\lambda_1} = N(T-\lambda_1 I) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

So, E_{λ_1} is the solution space of the system of linear equations

$$\begin{aligned} -x_1 \quad\quad + x_3 &= 0 \\ 2x_1 - 2x_2 + 2x_3 &= 0 \\ x_1 \quad\quad - x_3 &= 0 \end{aligned}$$

So, the system is $x_1 = x_3$

and $2x_1 - 2x_2 + 2x_3 = 0$

Putting $x_1 = x_3 = k$, $k \in \mathbb{R}$, we have

$2x_2 = 4k$ or, $x_2 = 2k$

So, we see $E_{\lambda_1} = \left\{ k\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} : k \in \mathbb{R} \right\}$

So, $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$ is a basis for E_{λ_1}

Similarly, $E_{\lambda_2} = N(T - \lambda_2 I)$ is the solution space of the system of linear equations

$$\begin{aligned} x_1 \quad + x_3 &= 0 \\ 2x_1 \quad + 2x_3 &= 0 \\ x_1 \quad + x_3 &= 0 \end{aligned}$$

So, the system is equivalent to

$$x_1 + x_3 = 0$$

So, let $x_1 = -x_3 = k_1$ and $x_2 = k_2$,

k_1, k_2 are arbitrary real numbers

So, $E_{\lambda_2} = \left\{ \begin{pmatrix} k_1 \\ k_2 \\ -k_1 \end{pmatrix} : k_1, k_2 \in \mathbb{R} \right\}$

$$= \left\{ k_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : k_1, k_2 \in \mathbb{R} \right\}$$

So, it follows that $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis

for E_{λ_2} and $\dim(E_{\lambda_2}) = 2$. Observe that union of two bases just derived, namely

$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is linearly

independent and hence a basis for $\mathbb{R}^3$.

This basis consists of the eigenvectors of T.

Consequently, T is diagonalizable.